Master's thesis

On the l^2 decoupling theorem

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The purpose of this thesis is to act as a guide for the 2017 article *A study guide for the l^2 decoupling theorem* by J. Bourgain and C. Demeter. However, this thesis is self-sufficient. The aim has been to give a detailed presentation and handle the weight exponent E especially carefully in the arguments.

We begin by presenting the decoupling inequality of the l^2 decoupling theorem and the associated Fourier transform -like operator. The theorem concerns finding a satisfactory upper bound for the decoupling constant related to the inequality. We also list some general results that a graduate student might not be very familiar with; among them are a few consequences of Hölder's inequality. We move on to study the properties of the weight functions that we use in the L^p-norms in the decoupling. We present two operator lemmas to which we can reduce many of our arguments. The other lemma gives us the opportunity to use certain Schwartz functions in our proofs. We then move on to prove the l^2 decoupling theorem in the lower range $2 \leq p \leq \frac{2n}{n-1}$. This includes the definition of multilinear decoupling constants and an iterative process.

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Chapter 1

Introduction

1.1 Preface

This thesis follows the article [3] by Jean Bourgain and Ciprian Demeter called *A Study Guide for the l^2 Decoupling Theorem.* However, this text is self-sufficient, that is, you do not need the original article to be able to follow along. This text acts as a "study guide for a study guide", presenting techniques that are used in decoupling. An attempt has been made to present a lot of details. The purpose of this is to help beginners get to grips with this subject. Having said that, we (unfortunately) take some essential results as granted (see chapter 6), which will leave some things in the dark. For instance, the details of the role of ν-transverse cubes in the argument will not be examined. Also, the reason why exactly choosing the weight exponent E to be greater than $100n$ suffices is not revealed.

The contributions of this thesis in addition to an overall detailed exposition include: the careful treatment of the exponent E, especially in Proposition 5.2.3, and construction of the Schwartz functions to be used in proofs. Also, the operator lemmas in Section 4.2 are presented in detail in both cases and have received some layout modifications.

What is l^2 decoupling? It refers to a collection of inequalities where on the left-hand side there is an L^p norm ($p \geq 2$) of a function f and on the right-hand side there is a l^2 norm of a (possibly infinite) sequence of L^p norms of functions that depend on f in a specific way. Usually, the functions on the right-hand side are so called Fourier projections generated from f that are associated with pairwise disjoint sets. The right-hand side is multiplied by a large enough constant for the inequality to hold; this constant is called a decoupling constant. The topic of this thesis concerns one specific case of l^2 decoupling, where the disjoint sets form a partition of a cube in $\mathbb{R}^{n-1}$. *Decoupling* concerns more than just l^2 decoupling. A recent book by C. Demeter, published in 2020, that covers decoupling in a wide generality is [7].

The realm of decoupling was initiated by T. Wolff in [13] in 2000. Since then, new techniques involved in decoupling and new applications of decoupling have been discovered, for instance in the milestone paper [2] (2013) by J. Bourgain. In [6], it is l^2 decoupling that Bourgain, Demeter and Guth use to prove the famous number-theoretic problem called Vinogradov's mean value theorem. The theorem concerns finding an upper bound for the amount of integer solutions to a specific equation system. Other recent works include [5] (the paper that [3] is a study guide to), [9] (a short proof of l^2 decoupling), [11] (about Vinogradov's mean value theorem) as well as [1] (a connection to Riemann zeta function) and [4] (an extension of [5]).

This thesis begins with a presentation of the Fourier transform -like operator E_Q. Then we introduce the weight functions that we use as weights in L^p-norms of the functions produced by the operator E_Q. The main result that considers the so-called decoupling constant is given in the first chapter as well.

The second chapter is devoted to arming ourselves with useful theorems from the realm of real analysis. Schwartz functions are defined along with some results. We will use some specific Schwartz functions to aid us in proving some of the propositions later on.

In the third chapter we examine some properties of the weight functions and develop a useful operator lemma. We can reduce many of the arguments that we will face to this lemma.

In the fourth chapter we prove propositions that describe decoupling in L^2 and how the decoupling constant is related to smaller cubes as well.

In chapter five, we give the definition of a decoupling constant that depends on some additional parameters and present a couple of results from [3]. Among these is a result that gives the basis for the induction used in the final chapter.

The sixth chapter is about the quantities A_p and D_p. We develop inequalities that concern these quantities and that will be of use in the final chapter.

The seventh and final chapter is devoted to the proof of the main result of this thesis.

I would like to express my gratitude to my thesis supervisor Tuomas Hytönen for all the guidance he has kindly offered to me.

1.2 Notations

$\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$.

Throughout this text $n \in \mathbb{N}_+$. Often we restrict n to be greater or equal to 2. Whenever the Euclidean space $\mathbb{R}^{n-1}$ is brought up, we implicitly assume that $n \geq 2$.

If we mention measurability (e.g. in the form of measurable functions, measurable subsets, integrals) and do not explicitly present the measure in question, we consider the Lebesgue-measure in $\mathbb{R}^n$. The symbol m_n is also used to refer to the Lebesgue measure in $\mathbb{R}^n$.

The symbol p is reserved for the power in L^p spaces. Throughout, $p < \infty$.

The operation $a \cdot \infty$, where $0 < a < \infty$, might occur in formulas. We agree that $a \cdot \infty = \infty$, if $a > 0$. We agree that $0 \cdot \infty = 0$.

For a Lebesgue measurable set $A \subset \mathbb{R}^n$, we use the standard notation

$$L^p(A) = \{f : A \to \mathbb{C} \text{ Lebesgue measurable} \mid \int |f|^p \, \mathrm{d}m_n < \infty\}.$$

Cubes in $\mathbb{R}^n$ are in a big role in this thesis. **Throughout, all cubes have side length in $2^{\mathbb{Z}} := \{2^z | z \in \mathbb{Z}\}$!!** We do this restriction to make it possible to create unique partitions of cubes into smaller cubes. By "all cubes" we mean the cubes in the statements of theorems, propositions, lemmas and definitions that are usually denoted by B, Q, q, Δ. However, in proofs we might construct auxiliary cubes with arbitrary side lengths.

Because of the nature of integration, the inclusion (or exclusion) of the boundaries of cubes does not essentially matter since the boundaries are null sets. With this in mind, we decide to present the cubes as *closed*, that is, including their boundaries. A consequence of this is that when we "partition" bigger cubes into smaller cubes, we do not *strictly* make a partition; we allow the boundaries of the smaller cubes to overlap (see notations for cubes and partitions in section 2.3).

Chapter 2

The beginning

In this chapter, we present the principal result of the thesis, the l^2 decoupling theorem.

2.1 The operator E_Q

- Here $n \geq 2$.
- We denote $e(t) := \mathrm{e}^{\mathrm{i}2\pi t}$ for each $t \in \mathbb{R}$.
- Let $Q \subset [0,1]^{n-1}$ be a cube. We let $g : Q \to \mathbb{C}$ be Lebesgue-measurable (the dimension is $n-1$) i.e. the g-preimage of an open set is measurable w.r.t. the Lebesgue measure. Additionally, let us suppose that the integral $\int_Q |g| \,\mathrm{d}\xi$ is finite. In short, we assume that $g \in L^1(Q)$.
- Assume that $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Since we assume that g is integrable, also

  $$\begin{aligned} \xi \mapsto\ & g(\xi)e(\xi_1 x_1 + \ldots + \xi_{n-1}x_{n-1} + (\xi_1^2 + \ldots \xi_{n-1}^2)x_n) \\ &= g(\xi)e(\xi_1 x_1 + \ldots + \xi_{n-1}x_{n-1} + |\xi|^2 x_n) \end{aligned}$$

  is integrable on Q. This is because the latter function is measurable as a product of measurable functions and $|g(\xi)e(\xi_1 x_1 + \ldots + \xi_{n-1}x_{n-1} + |\xi|^2 x_n)| = |g(\xi)|$.
- We define the function $E_Q g : \mathbb{R}^n \to \mathbb{C}$ as follows:

  $$E_Q g(x) = \int_Q g(\xi)e(\xi_1 x_1 + \ldots + \xi_{n-1}x_{n-1} + |\xi|^2 x_n) \,\mathrm{d}\xi,$$

  where $\xi = (\xi_1, \ldots, \xi_{n-1})$ and $x = (x_1, \ldots, x_n)$.

- **Claim: $E_Q g : \mathbb{R}^n \to \mathbb{C}$ is continuous.** Observe that

  (1) $|g(\xi)e(\xi_1 x_1 + \ldots + \xi_{n-1}x_{n-1} + |\xi|^2 x_n)| \le |g(\xi)|$ for all $\xi \in Q$ and $x \in \mathbb{R}^n$,

  (2) $|g| \in L^1(Q)$,

  (3) for a fixed $\xi \in \mathbb{R}^{n-1}$ the function $x \mapsto e(\xi_1 x_1 + \ldots + \xi_{n-1}x_{n-1} + |\xi|^2 x_n)$ is a continuous function from $\mathbb{R}^n$ into $\mathbb{C}$.

  Hence if $x_k \to a$, $k \to \infty$, in $\mathbb{R}^n$, then by Lebesgue's dominated convergence theorem
  $$\lim_{k\to\infty} E_Q g(x_k) = E_Q g(\lim_{k\to\infty} x_k) = E_Q g(a).$$
  This proves the continuity.

- Especially $E_Q g : \mathbb{R}^n \to \mathbb{C}$ is a measurable function, and for each $x \in \mathbb{R}^n$
  $$\begin{aligned} |E_Q g(x)| &\le \int_Q \left| g(\xi)e(\xi_1 x_1 + \ldots + \xi_{n-1}x_{n-1} + |\xi|^2 x_n)\right| \mathrm{d}\xi \\ &= \int_Q |g(\xi)| \,\mathrm{d}\xi \\ &= \|g\|_{L^1(Q)} < \infty. \end{aligned}$$
  Thus $E_Q g$ is a bounded function and $E_Q g \in L^\infty(\mathbb{R}^n)$.

- We write Eg as a shorthand for $E_{[0,1]^{n-1}} g$.

2.2 How to interpret $E_Q g$ as a Fourier transform in $\mathbb{R}^n$

First, we recall the definition of Fourier transform in $\mathbb{R}^n$ with respect to Lebesgue measure.

Definition 2.2.1. Let $f \in L^1(\mathbb{R}^n)$. Then the *Fourier transform* $\widehat{f}$ of f is defined as the function
$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x)\mathrm{e}^{-\mathrm{i}2\pi\xi\cdot x} \,\mathrm{d}x,$$
where $\xi \in \mathbb{R}^n$. Here we integrate using Lebesgue measure.

Respectively, the *inverse Fourier transform* $\mathcal{F}^{-1}f$ of f is defined by
$$(\mathcal{F}^{-1}f)(\xi) := \int_{\mathbb{R}^n} f(x)\mathrm{e}^{\mathrm{i}2\pi\xi\cdot x} \,\mathrm{d}x.$$

- **Remark:** The Fourier transform $\widehat{f} : \mathbb{R}^n \to \mathbb{C}$ is continuous. This can be proved using Lebesgue's dominated convergence theorem. Furthermore, the so-called *Riemann-Lebesgue lemma* states that $\widehat{f}(x) \to 0$, when $|x| \to \infty$. These together imply the result that $\widehat{f} : \mathbb{R}^n \to \mathbb{C}$ is in fact *uniformly* continuous.

- **Remark:** There are several conventions (e.g. 2π removed from the exponent) to the definition of the Fourier transform; above we used one of them. However, all of the conventions carry the same essential properties.

Here we only consider the case $n = 2$.

Let $\mathbb{P} = \{(\xi, \xi^2) : 0 \leq \xi \leq 1\}$. Then $\mathbb{P}$ can be called a truncated paraboloid. Fix an interval $Q \subset [0,1]$ (open, closed, half-open) and an *integrable* function $g : Q \to \mathbb{C}$.

As before, let E_Q be an operator that defines a function $E_Q g : \mathbb{R}^2 \to \mathbb{C}$ with the formula

$$E_Q g(x) = \int_Q g(\xi_1) e(\xi_1 x_1 + \xi_1^2 x_2) \, \mathrm{d}m_1(\xi_1),$$

where $e(t) = \mathrm{e}^{2\pi \mathrm{i} t}$, $x = (x_1, x_2) \in \mathbb{R}^2$ and integration is with respect to the Lebesgue measure m_1 in $\mathbb{R}$.

In the Bourgain-Demeter paper the above integral is given an interpretation as "the Fourier transform $\widehat{gd\sigma}$, where the measure $d\sigma$ is the lift of the Lebesgue measure from $[0,1]$ to the paraboloid".

Aside from a few differences, the formula for $E_Q g$ does look like a formula for the Fourier transform of g. Observe that the integration variable ξ_1 has a dimension one smaller than x. Inside the exponential function this is compensated by adding ξ_1^2 as a factor for x_2. Together they form a point (ξ_1, ξ_1^2) on the paraboloid $\mathbb{P}$.

In fact, $E_Q g$ is a kind of a Fourier transform of a function that depends considerably on g. *Let us formally show this.*

First, let us construct the measure μ that acts as "the lift of the Lebesgue measure from $[0,1]$". Define $\rho : \mathbb{R} \to \mathbb{R}^2$ with $\rho(t) = (t, t^2)$. Let the function $\mu : \mathrm{Bor}\, \mathbb{R}^2 \to [0, +\infty]$ be defined as follows (we will show that it is well-defined):

$$\mu(E) = m_1(\rho^{-1} E),$$

where m_1 is the Lebesgue measure in $\mathbb{R}$. In other words, the measure μ is the 1-dimensional Lebesgue measure of the first projection of a Borel set.

Lemma 2.2.2. *The function $\mu : \mathrm{Bor}\, \mathbb{R}^2 \to [0, +\infty]$ is a well-defined measure and hence we have the measure space $(\mathbb{R}^2, \mathrm{Bor}\, \mathbb{R}^2, \mu)$.*

Proof. (Note especially the bolded part.) Let $E \in \mathrm{Bor}\, \mathbb{R}^2$. We know that $\rho : \mathbb{R} \to \mathbb{R}^2$ is a Borel function because it is continuous. **Thus $\rho^{-1}E$ is a Borel set** (because it is a preimage of a Borel set in a Borel function). Hence $\pi^{-1}E$ is Lebesgue-measurable in $\mathbb{R}$. We have shown that μ is a well-defined function.

Clearly $\mu(\emptyset) = 0$. Let $E_j \in \text{Bor}\,\mathbb{R}^2$ be disjoint sets, where $j \in \mathbb{N}$. Now

$$\mu(\bigcup_j E_j) = m_1(\rho^{-1}(\bigcup_j E_j)) = m_1(\bigcup_j \rho^{-1}E_j) = \sum_j m_1(\rho^{-1}E_j) = \sum_j \mu(E_j),$$

since the sets $\rho^{-1}E_j$ are disjoint. Thus μ is countably additive. We have shown that μ is a measure. $\square$

In what follows we will assume that $x = (x_1, x_2) \in \mathbb{R}^2$ is fixed. Using ρ we can write

$$\begin{aligned} E_Q g(x) &= \int_Q g(\xi_1) e(\rho(\xi_1) \cdot x) \, \mathrm{d}m_1(\xi_1) \\ &= \int_Q (g \circ \rho^{-1})(\rho(\xi_1)) e(\rho(\xi_1) \cdot x) \, \mathrm{d}m_1(\xi_1). \end{aligned}$$

Observe that ρ is a bijection $[0,1] \to \mathbb{P}$ and $\rho^{-1} : \mathbb{P} \to [0,1]$ is its inverse. We have $\rho^{-1}((s,t)) = s$ for all $(s,t) \in \mathbb{P}$.

We let 1_A denote the indicator function of an arbitrary set A. Moreover, we *make an agreement* concerning the product function (notation) $f1_A$, where $f : A \to \mathbb{C}$ is a function and $A \subset X$. Namely, we define the function $f1_A : X \to \mathbb{C}$ as follows:

- $(f1_A)(t) := f(t)1_A(t) = f(t)$, when $t \in A$,
- $(f1_A)(t) := 0$, when $t \in X \setminus A$.

The following observation (1) for indicator functions creates a question whether something similar would hold for all Borel measurable functions. Clearly $\rho Q \in \text{Bor}\,\mathbb{R}^2$, since it is a connected "piece" of the paraboloid $\mathbb{P}$. Hence we can integrate over ρQ with respect to μ.

(1) Let $E \in \text{Bor}\,\mathbb{R}^2$. Now

$$\begin{aligned} \int_{\rho Q} 1_E(\xi) \, \mathrm{d}\mu(\xi) &= \int_{\mathbb{R}^2} 1_{\rho Q \cap E}(\xi) \, \mathrm{d}\mu(\xi) \\ &= \mu(\rho Q \cap E) \\ &= m_1(\rho^{-1}(\rho Q \cap E)) \\ &= m_1(Q \cap \rho^{-1}E) \qquad (\rho \text{ is an injection}) \\ &= \int_{\mathbb{R}} 1_{Q \cap \rho^{-1}E}(\xi_1) \, \mathrm{d}m_1(\xi_1) \\ &= \int_Q 1_{\rho^{-1}E}(\xi_1) \, \mathrm{d}m_1(\xi_1) \\ &= \int_Q 1_E(\rho(\xi_1)) \, \mathrm{d}m_1(\xi_1). \end{aligned}$$

Let us generalize (1) further.

(2) Let $s\colon \mathbb{R}^2 \to \mathbb{R}$ be a simple Borel measurable function defined by $s = \sum_{j=1}^k a_j 1_{A_j}$, where $a_j \in \mathbb{R}, a_j \geq 0$ and $A_j \in \operatorname{Bor} \mathbb{R}^2$ are distinct sets. Using (1),

$$\begin{aligned}
\int_{\rho Q} s \,\mathrm{d}\mu &= \int_{\rho Q} \sum_{j=1}^k a_j 1_{A_j} \,\mathrm{d}\mu \\
&= \sum_{j=1}^k a_j \int_{\rho Q} 1_{A_j} \,\mathrm{d}\mu \\
&= \sum_{j=1}^k a_j \int_Q (1_{A_j} \circ \rho) \,\mathrm{d}m_1 && \text{by (1)} \\
&= \int_Q \sum_{j=1}^k a_j (1_{A_j} \circ \rho) \,\mathrm{d}m_1 \\
&= \int_Q (s \circ \rho) \,\mathrm{d}m_1.
\end{aligned}$$

(3) Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f \geq 0$, be a non-negative Borel measurable function. Now there exists a sequence $(s_k)_{k \in \mathbb{N}_+}$ of simple functions as in (2) such that $0 \leq s_1 \leq s_2 \leq \ldots$ and $\lim_{k\to\infty} s_k(x) = f(x)$ for all $x \in \mathbb{R}^2$ (see e.g [12] Theorem 1.17). Using monotone convergence and (2),

$$\begin{aligned}
\int_{\rho Q} f \,\mathrm{d}\mu &= \int_{\rho Q} \lim_{k\to\infty} s_k \,\mathrm{d}\mu \\
&= \lim_{k\to\infty} \int_{\rho Q} s_k \,\mathrm{d}\mu && \text{(monotone convergence)} \\
&= \lim_{k\to\infty} \int_Q (s_k \circ \rho) \,\mathrm{d}m_1 && \text{by (2)} \\
&= \int_Q \lim_{k\to\infty} (s_k \circ \rho) \,\mathrm{d}m_1 && \text{(monotone convergence)} \\
&= \int_Q (f \circ \rho) \,\mathrm{d}m_1.
\end{aligned}$$

(4) Let $\gamma = \mathbb{R}^2 \to \mathbb{C}$ be a Borel-measurable function satisfying $\int_{\mathbb{R}^2} |\gamma| \,\mathrm{d}\mu < \infty$. Let $u = \operatorname{Re} \gamma$ and $v = \operatorname{Im} \gamma$. Using (3),

$$\begin{aligned}
\int_{\rho Q} \gamma \,\mathrm{d}\mu &= \int_{\rho Q} u^+ \,\mathrm{d}\mu - \int_{\rho Q} u^- \,\mathrm{d}\mu + \mathrm{i} \int_{\rho Q} v^+ \,\mathrm{d}\mu - \mathrm{i} \int_{\rho Q} v^- \,\mathrm{d}\mu \\
&= \int_Q (u^+ \circ \rho) \,\mathrm{d}m_1 - \int_Q (u^- \circ \rho) \,\mathrm{d}m_1 + \mathrm{i} \int_Q (v^+ \circ \rho) \,\mathrm{d}m_1 - \mathrm{i} \int_Q (v^- \circ \rho) \,\mathrm{d}m_1 \\
&= \int_Q (u \circ \rho)^+ \,\mathrm{d}m_1 - \int_Q (u \circ \rho)^- \,\mathrm{d}m_1 + \mathrm{i} \int_Q (v \circ \rho)^+ \,\mathrm{d}m_1 - \mathrm{i} \int_Q (v \circ \rho)^- \,\mathrm{d}m_1 \\
&= \int_Q (\gamma \circ \rho) \,\mathrm{d}m_1.
\end{aligned}$$

To use (4) for our needs, we need a Borel function. It turns out that it is beneficial to choose a Borel function $g^* : Q \to \mathbb{C}$ such that $g^*(t) = g(t)$ for m_1-almost every $t \in Q$. It is known that such a Borel function exists; see e.g. Lemma 1 of Theorem 8.12 in [12]. Then, continuing from where we left off,

$$\begin{aligned} E_Q g(x) &= \int_Q g^*(\xi_1) e(\rho(\xi_1) \cdot x) \,\mathrm{d}m_1(\xi_1) \\ &= \int_Q (g^* \circ \rho^{-1})(\rho(\xi_1)) e(\rho(\xi_1) \cdot x) \,\mathrm{d}m_1(\xi_1). \end{aligned}$$

Let us define

$$\gamma(\xi) := g^*(\rho^{-1}(\xi)) e(\xi \cdot x),$$

for $\xi \in \rho Q$.

Lemma 2.2.3. *The function $\gamma : \rho Q \to \mathbb{C}$ is μ-measurable.*

Proof. Clearly $\xi \mapsto e(\xi \cdot x)$ is a continuous function $\rho Q \to \mathbb{C}$. Hence it is a Borel function. For the same reason, $\rho^{-1} : \rho Q \to Q$ is a Borel function and hence $g^* \circ \rho^{-1} : \rho Q \to \mathbb{C}$ is a Borel function. A product of Borel functions is a Borel function. This proves the claim. □

We see that

$$\begin{aligned} \int_{\rho Q} |\gamma| \,\mathrm{d}\mu &= \int_{\rho Q} |g^*(\rho^{-1}(\xi)) e(\xi \cdot x)| \,\mathrm{d}\mu(\xi) \\ &= \int_{\rho Q} |g^* \circ \rho^{-1}| \,\mathrm{d}\mu \\ &= \int_Q |g^*| \,\mathrm{d}m_1 && \text{by (3)} \\ &= \int_Q |g| \,\mathrm{d}m_1 \\ &< \infty. \end{aligned}$$

Thus by (4)

$$\begin{aligned} E_Q g(x) &= \int_Q \gamma(\rho(\xi_1)) \,\mathrm{d}m_1(\xi_1) \\ &= \int_{\rho Q} \gamma(\xi) \,\mathrm{d}\mu(\xi) \\ &= \int_{\mathbb{R}^2} ((g^* \circ \rho^{-1}) 1_{\rho Q})(\xi) e(\xi \cdot x) \,\mathrm{d}\mu(\xi). \end{aligned}$$

From the last formula we read that $E_Q g$ is the inverse μ-Fourier transform of the function

$$(g^* \circ \rho^{-1}) 1_{\rho Q}.$$

(The μ-Fourier transform is defined analogously to the case of Lebesgue measure.) That is, we integrate using the lower dimensional Lebesgue measure in the projection of the paraboloid.

In the light of the sign of the exponent in the formula for a Fourier transform not mattering much, Fourier transform and inverse Fourier transform are practically the same. Because of this and by using a convention for $\mathrm{d}\mu$ we could also say that

$$E_Q g \text{ is the Fourier transform of } ((g^* \circ \rho^{-1})1_{\rho Q})\,\mathrm{d}\mu.$$

2.3 Notation for cubes, the partition $\mathrm{Part}_\alpha(B)$ and implicit constants

We will write $B = B(c_B, R)$ for the closed cube in $\mathbb{R}^n$ centered at c_B and with side length $R > 0$.

We use l to denote the side length of a cube e.g. $l(Q) = 2$ means the side length of a cube Q is 2.

Given a cube $B \subset \mathbb{R}^n$ with side length $l(B) \in 2^{\mathbb{Z}}$ and $\alpha \in 2^{\mathbb{Z}}$ such that $\alpha \leq l(B)$, we will denote by $\mathrm{Part}_\alpha(B)$ the unique *essential* partition of B by using closed cubes B_α of side length α. By *essential* we mean that the boundaries of the cubes B_α are allowed to overlap and an inner point of a cube $B\alpha$ is not contained in any other partition cube. It is easy to believe (e.g. by drawing a picture of an example case) that the essential partition exists and that it is unique. Throughout, we will refer to $\mathrm{Part}_\alpha(B)$ as a partition, leaving the word essential out for brevity's sake.

Because the boundaries of cubes have Lebesgue measure zero, the boundaries are negligible in calculating the L^p-norms of the functions $E_Q g$. Hence thinking about the cubes Q as closed is a natural and simple way of interpreting the l^2 decoupling theorem (see section 2.5).

(Alternatively, we could think of the cubes as having a suitable amount of boundary (e.g. half open cubes), so that we would get *actual* partitions.)

Throughout this paper we use the following notation: Let F and G be non-negative real-valued (sometimes also $+\infty$ is allowed as a value) variables that share some parameter domain $P = \{v_1, \ldots, v_m\}$ (in practice the domain is implicitly known in the context) i.e. the values of F and G depend on the same parameters. We denote $F \lesssim G$ if there exists a *strictly positive* constant $0 < C < \infty$ such that $F \leq CG$ in the domain P. We denote

$$F \lesssim_{v_{j_1},\ldots,v_{j_k}} G \tag{2.1}$$

if the *only* parameters that the value of the constant $0 < C < \infty$ depends on are $\upsilon_{j_1}, \ldots, \upsilon_{j_k}$, where $1 \leq j_1, \ldots, j_k \leq m$. The constant C associated with the $\lesssim$-notation is called *implicit*.

An example: Let $F, G : [0,1] \to \mathbb{R}$ be functions. If there exists a constant $C > 0$ such that the inequality $F(x) \leq CG(x)$ holds for all $x \in [0,1]$, we can write $F \lesssim G$. One can say that here the expression $F \lesssim G$ means that if for some x it holds that $G(x)$ is strictly smaller than $F(x)$, then $G(x)$ is smaller only by a predetermined constant that does not depend on x.

We denote $F \sim G$, if $F \lesssim G$ and $G \lesssim F$ (with parameters that at least one of the constants depends on added as subscripts if necessary).

2.4 The weight function

Definition 2.4.1. Let $B = B(c, R) \subset \mathbb{R}^n$ be a cube, where $R > 0$. Let $E > n$.

We define a *weight function* $w_{B,E}$ by

$$w_{B,E}(x) := \frac{1}{\left(1 + \frac{|x-c|}{R}\right)^E}$$

for $x \in \mathbb{R}^n$. Throughout, we will write $w_B = w_{B,E}$ just to simplify formulas in cases where only weight functions with the same exponent E are present.

The function $w_{B,E}$ is a sort of a smooth version of the indicator function 1_B (i.e. without the jump from 0 to 1 near the boundary of the set). This version gets values in $(0, 1]$, attains the value 1 only in the center of the cube, and decreases the further away from the center of the cube it is evaluated. Observe that in integrating $w_{B,E}$ over $\mathbb{R}^n$ we have to consider the values outside B, unlike in the case of integrating the function 1_B. The graph of $w_{B,E}$ is, you could say, round-shaped (instead of having sharp edges like a cube). The absolute value in the definition above is the Euclidean norm.

Later in the thesis we will only consider large exponents $E \geq 100n$ in the denominator of $w_{B,E}$. The exponent is ”chosen large enough to guarantee various integrability requirements”, as Bourgain and Demeter state in their paper. Lillian Pierce [11] states that: ”The exponent E is simply chosen large enough that we may apply e.g. Hölder’s inequality as many times as the argument requires, and still yield a weight, which we will also call w_B, that is tailored to the ball B and decays sufficiently to be integrable in $\mathbb{R}^n$”. The condition $E > n$ guarantees that $w_{B,E}$ is integrable, as is seen shortly. Many results in this thesis are presented for all $E > n$.

We will make use of the following fundamental estimate of the magnitude of the L^1 norm of a weight function $w_{B,E}$. In particular, this proves that $w_{B,E} \in L^1(\mathbb{R}^n)$.

Lemma 2.4.2. *Let $n \geq 1$ and $E > n$. Then the following holds:*

$$\int_{\mathbb{R}^n} w_{B,E}(x)\,\mathrm{d}x \ \sim_{E,n} \ R^n$$

for all cubes $B = B(c,R)$ in $\mathbb{R}^n$, $R > 0$.

Proof.

$$\int_{\mathbb{R}^n} w_{B,E}(x) = \int_{|x-c|\leq R} \Big(1+\frac{|x-c|}{R}\Big)^{-E} \mathrm{d}x + \sum_{k=1}^{\infty} \int_{2^{k-1}R<|x-c|\leq 2^k R} \Big(1+\frac{|x-c|}{R}\Big)^{-E} \mathrm{d}x,$$

where

$$\int_{|x-c|\leq R} \underbrace{\Big(1+\frac{|x-c|}{R}\Big)^{-E}}_{\leq 1} \mathrm{d}x \lesssim_n R^n$$

and

$$\begin{aligned}
\sum_{k=1}^{\infty} \int_{2^{k-1}R<|x-c|\leq 2^k R} \Big(1+\frac{|x-c|}{R}\Big)^{-E} \mathrm{d}x &\leq \sum_{k=1}^{\infty} \int_{|x-c|\leq 2^k R} \underbrace{(1+2^{k-1})^{-E}}_{\leq (2^{k-1})^{-E}} \mathrm{d}x \\
&\lesssim_n \sum_{k=1}^{\infty} (2^k R)^n (2^{k-1})^{-E} \\
&\lesssim_E R^n \sum_{k=1}^{\infty} (2^{n-E})^k \\
&\lesssim_{E,n} R^n,
\end{aligned}$$

where the geometric series converges because $n - E < 0$. The claim

$$\int_{\mathbb{R}^n} w_{B,E}(x)\,\mathrm{d}x \ \lesssim_{E,n} \ R^n$$

follows by taking the sum of the two implicit constants.

For the other direction,

$$R^n \lesssim_E 2^{-E} R^n \lesssim_n \int_{|x-c|<R} 2^{-E}\,\mathrm{d}x \leq \int_{|x-c|<R} \Big(1+\frac{|x-c|}{R}\Big)^{-E} \mathrm{d}x \leq \int_{\mathbb{R}^n} w_{B,E}(x)\,\mathrm{d}x.$$

$\square$

For $p \geq 1$ and non-negative $v \in L^1(\mathbb{R}^n)$ we denote

$$L^p(v) := \Big\{ f : \mathbb{R}^n \to \mathbb{C} \text{ measurable} : \int_{\mathbb{R}^n} |f(x)|^p v(x)\,\mathrm{d}x < \infty \Big\}.$$

For $f \in L^p(v)$ we define

$$\|f\|_{L^p(v)} := \Big(\int_{\mathbb{R}^n} |f(x)|^p v(x)\,\mathrm{d}x\Big)^{\frac{1}{p}}.$$

The norm here is an example of a *weighted L^p norm* (the *weight* in this case is v).

The L^p norms we will consider are finite:

Lemma 2.4.3. *Let $p \geq 1$. Let $f : \mathbb{R}^n \to \mathbb{C}$ be a bounded function and $v \in L^1(\mathbb{R}^n)$ be a non-negative function. Then $\|f\|_{L^p(v)} < \infty$.*

Proof.

$$\int_{\mathbb{R}^n} |f(x)|^p v(x)\,\mathrm{d}x \leq \|f\|_{L^\infty}^p \int_{\mathbb{R}^n} v(x)\,\mathrm{d}x < \infty.$$

□

2.5 The l^2 decoupling theorem

In the Main Theorem 2.5.3 only the weight exponent $E = 100n$ will be considered, since it is large enough for 2.5.3 to hold. That is why in this section we consider the exponent $100n$. However, in order to prove Theorem 2.5.3 we will also need knowledge about exponents strictly bigger than $100n$. This can be seen for example in Theorem 6.2.1.

Let $n \geq 2$. Fix a cube $B = B(c_B, R) \subset \mathbb{R}^n$, $g : [0,1]^{n-1} \to \mathbb{C}$ and a cube $Q \subset [0,1]^{n-1}$. Let us break down the mathematical entity of

$$\begin{aligned}
\|E_Q g\|_{L^p(w_B)} &= \Big(\int_{\mathbb{R}^n} |E_Q g(x)|^p w_B(x)\,\mathrm{d}x\Big)^{\frac{1}{p}} \\
&= \Big(\int_{\mathbb{R}^n} |w_B(x)^{\frac{1}{p}} E_Q g(x)|^p\,\mathrm{d}x\Big)^{\frac{1}{p}} \\
&= \|w_B^{\frac{1}{p}} E_Q g\|_{L^p(\mathbb{R}^n)},
\end{aligned}$$

where $2 \leq p < \infty$ and

$$w_B(x) = w_{B,100n}(x).$$

Now $E_Q g$ is the intricate inverse Fourier transform described in section 2.2. We only use the values of g in the cube Q to produce the transform. Note that $w_B \in L^1(\mathbb{R}^n)$ by Lemma 2.4.2. Since additionally $E_Q g$ is bounded, it follows from Lemma 2.4.3 that $\|E_Q g\|_{L^p(w_B)} < \infty$.

We weight $E_Q g$ with the pth root of w_B. Remember that w_B achieves values in $(0,1]$ so this procedure dampens the function. The least dampening happens near the cube B and the most dampening happens far away from B. Taking the

pth root of w_B makes the dampening a bit weaker; the bigger p is, the weaker the dampening is. Actually, taking the pth root will not affect the weight function itself arbitrarily much, since we restrict $2 \leq p \leq \frac{2(n+1)}{n-1} \leq 6$ in Theorem 2.5.3. (Observe that the upper bound for p is 6, when $n \geq 2$. The upper bound decreases, when n increases. When $n = 5$, the upper bound is 3.)

Then we take the L^p norm of this dampened function. And voilà!

Definition 2.5.1. (**Decoupling constant**). Let $n \geq 2$, $p \geq 2$ and $\delta \in 4^{-\mathbb{N}}$. We define the decoupling constant $\mathrm{Dec}_n(\delta, p)$ to be the smallest non-negative real number such that

$$\|Eg\|_{L^p(w_B)} \leq \mathrm{Dec}_n(\delta, p) \left(\sum_{Q \in \mathrm{Part}_{\delta^{\frac{1}{2}}}([0,1]^{n-1})} \|E_Q g\|^2_{L^p(w_B)} \right)^{\frac{1}{2}}$$

holds for every cube $B \subset \mathbb{R}^n$ with side length δ^{-1} and every $g : [0,1] \to \mathbb{C}$.

Lemma 2.5.2. *The decoupling constant is well defined.*

Proof. We could prove this already. However, we will present the proof in Lemma 2.6.1 in a more general setting. □

The inequality

$$\|Eg\|_{L^p(w_B)} \leq \mathrm{Dec}_n(\delta, p) \left(\sum_{Q \in \mathrm{Part}_{\delta^{\frac{1}{2}}}([0,1]^{n-1})} \|E_Q g\|^2_{L^p(w_B)} \right)^{\frac{1}{2}},$$

can be seen as a modified inequality version of the Pythagorean theorem. Maybe this is seen clearer if both sides are first squared as in

$$\|Eg\|^2_{L^p(w_B)} \leq \mathrm{Dec}_n(\delta, p)^2 \sum_{Q \in \mathrm{Part}_{\delta^{\frac{1}{2}}}([0,1]^{n-1})} \|E_Q g\|^2_{L^p(w_B)}.$$

Observe that since we have an essential partition, we can write the left-hand side differently and get

$$\| \sum_{Q \in \mathrm{Part}_{\delta^{\frac{1}{2}}}([0,1]^{n-1})} E_Q g\|^2_{L^p(w_B)} \leq \mathrm{Dec}_n(\delta, p)^2 \sum_{Q \in \mathrm{Part}_{\delta^{\frac{1}{2}}}([0,1]^{n-1})} \|E_Q g\|^2_{L^p(w_B)}. \qquad (2.2)$$

On the left-hand side we integrate over the whole interval $[0,1]^{n-1}$ once and square the result. On the right-hand side we integrate once per each member Q of the partition and add the squares of these integrals together.

The following theorem is the main result of the thesis. We work our way towards proving this, which will ultimately happen in Chapter 8. We will only show the proof in the case $2 \leq p \leq \frac{2n}{n-1}$. Much of what is needed in the case $p > \frac{2n}{n-1}$ will be covered simultaneously, but the final arguments in that case can be found in [3].

Theorem 2.5.3. *Let us fix $n \in \mathbb{N}, n \geq 2$ and $2 \leq p \leq \frac{2(n+1)}{n-1}$. In addition, let us fix $\epsilon > 0$. Now there exists a constant $C > 0$ such that the following statement holds:*

$$\mathrm{Dec}_n(\delta, p) \leq C \cdot \delta^{-\epsilon}$$

for all $\delta \in 4^{-\mathbb{N}}$. That is,

$$\mathrm{Dec}_n(\delta, p) \lesssim_{\epsilon,p,n} \delta^{-\epsilon}$$

for all $\delta \in 4^{-\mathbb{N}}$.

Theorem 2.5.3 states that although the weighted L^p norms in question are not necessarily blessed with being "Pythagorean" in the sense that there is an inequality in (2.2), the inequality is not arbitrarily "wide"; we have an upper bound for the decoupling constant that depends on δ and the chosen exponent $-\epsilon$ of δ.

2.6 The decoupling constant $\mathrm{Dec}_n(\delta, p, E)$

We also define decoupling constants related to all other exponents $E > n$ with the same logic and prove the existence of the constants.

Let $n \geq 2$. Fix $p \geq 2$, $\delta \in 4^{-\mathbb{N}}$ and $E > n$.

Denote with $D_{\delta,p,E}$ the set of constants $0 \leq a < \infty$ that satisfy the inequality

$$\|Eg\|_{L^p(w_{B,E})} \leq a \cdot \Big(\sum_{Q \in \mathrm{Part}_{\delta^{\frac{1}{2}}}([0,1]^{n-1})} \|E_Q g\|^2_{L^p(w_{B,E})} \Big)^{\frac{1}{2}} \tag{2.3}$$

for every cube $B \subset \mathbb{R}^n$ with side length δ^{-1} and every $g \in L^1([0,1]^{n-1})$.

We show that $D_{\delta,p,E}$ has a smallest element, which we will call $\mathrm{Dec}_n(\delta, p, E)$.

We show first that the set $D_{\delta,p,E}$ is not empty.

Fix a cube $B \subset \mathbb{R}^n$ with side length δ^{-1} and $g\colon [0,1]^{n-1} \to \mathbb{C}$. Now

$$\begin{aligned}
\|Eg\|_{L^p(w_{B,E})} &= \Big\| \sum_{Q \in \mathrm{Part}_{\delta^{\frac{1}{2}}}([0,1]^{n-1})} E_Q g \Big\|_{L^p(w_{B,E})} \\
&\leq \sum_{Q \in \mathrm{Part}_{\delta^{\frac{1}{2}}}([0,1]^{n-1})} \|E_Q g\|_{L^p(w_{B,E})} && \text{(Minkowski)} \\
&\leq \Big(\sum_{Q \in \mathrm{Part}_{\delta^{\frac{1}{2}}}([0,1]^{n-1})} 1 \Big)^{\frac{1}{2}} \Big(\sum_{Q \in \mathrm{Part}_{\delta^{\frac{1}{2}}}([0,1]^{n-1})} \|E_Q g\|^2_{L^p(w_{B,E})} \Big)^{\frac{1}{2}} && \text{(Cauc.-Schw.)} \\
&= \delta^{-\frac{n-1}{4}} \Big(\sum_{Q \in \mathrm{Part}_{\delta^{\frac{1}{2}}}([0,1]^{n-1})} \|E_Q g\|^2_{L^p(w_{B,E})} \Big)^{\frac{1}{2}}. && (2.4)
\end{aligned}$$

Hence $\delta^{-\frac{n-1}{4}} \in D_{\delta,p,E}$.

But does $D_{\delta,p,E}$ have a smallest element? It turns out that it does.

We show that $d := \inf D_{\delta,p,E} \in D_{\delta,p,E}$. Clearly $d \geq 0$. If g and B are such that the right-hand side sum of (2.3) is zero, then so is the left-hand side, as can be seen in (2.4). If B and g are such that the right-hand side sum is non-zero, we get

$$\frac{\|Eg\|_{L^p(w_{B,E})}}{\left(\sum_{Q\in \mathrm{Part}_{\delta^{\frac{1}{2}}}([0,1]^{n-1})} \|E_Q g\|^2_{L^p(w_{B,E})}\right)^{\frac{1}{2}}} \leq a$$

for all $a \in D_{\delta,p,E}$, which implies that a above can be replaced with the infimum d. We showed that d is the smallest element of $D_{\delta,p,E}$.

Hence we proved the following:

Lemma 2.6.1. $\mathrm{Dec}_n(\delta, p, E)$*, as defined above, is a well defined non-negative number for* $p \geq 2$*,* $\delta \in 4^{-\mathbb{N}}$*,* $n \geq 2$ *and* $E > n$*. Additionally,*

$$\|Eg\|_{L^p(w_{B,E})} \leq \mathrm{Dec}_n(\delta, p, E)\Big(\sum_{Q\in \mathrm{Part}_{\delta^{\frac{1}{2}}}([0,1]^{n-1})} \|E_Q g\|^2_{L^p(w_{B,E})}\Big)^{\frac{1}{2}} \tag{2.5}$$

and $\mathrm{Dec}_n(\delta, p, E) \leq \delta^{-\frac{n-1}{4}}$*. In particular,* $\mathrm{Dec}_n(\delta, p) = \mathrm{Dec}_n(\delta, p, 100n)$*.*

Chapter 3

General preliminaries

In this chapter, we present and recall some general results that highlight the techniques we use to prove the l^2 decoupling theorem. These results will be used throughout. In particular, we define Schwartz functions and study the properties of them and their Fourier transforms.

3.1 The results

Theorem 3.1.1. *(*An inequality for averaged integrals*) Let* $1 \leq p \leq q < \infty$. *Let* (X, Γ, μ) *be a complete measure space such that* $0 < \mu(X) < \infty$. *If* $f : X \to \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ *is measurable and* $f \in L^q(X)$, *then*

$$\Big(\frac{1}{\mu(X)} \int_X |f|^p \,\mathrm{d}\mu\Big)^{\frac{1}{p}} \leq \Big(\frac{1}{\mu(X)} \int_X |f|^q \,\mathrm{d}\mu\Big)^{\frac{1}{q}}.$$

Proof. We can assume $p < q$.

Assume that $f \in L^q(X)$. Now

$$\int_X (|f|^p)^{\frac{q}{p}} \,\mathrm{d}\mu = \int_X |f|^q \,\mathrm{d}\mu < \infty.$$

Thus $|f|^p \in L^{\frac{q}{p}}(X)$.

Now $\frac{q}{p}, \frac{q}{q-p} > 1$. By Hölder's inequality

$$\int_X |f|^p \,\mathrm{d}\mu \leq \Big(\int_X |f|^q \,\mathrm{d}\mu\Big)^{\frac{p}{q}} \Big(\int_X 1^{\frac{q}{q-p}} \,\mathrm{d}\mu\Big)^{\frac{q-p}{q}} = \Big(\int_X |f|^q \,\mathrm{d}\mu\Big)^{\frac{p}{q}} \mu(X)^{\frac{q-p}{q}}.$$

By taking the pth root of both sides we get

$$\Big(\int_X |f|^p \,\mathrm{d}\mu\Big)^{\frac{1}{p}} \leq \Big(\int_X |f|^q \,\mathrm{d}\mu\Big)^{\frac{1}{q}} \mu(X)^{\frac{1}{p}} \mu(X)^{-\frac{1}{q}}$$

whence the claim follows by moving the terms around. $\square$

Theorem 3.1.2. *(*The n-product Hölder's inequality*) Let $n, K \in \mathbb{N}_+$. Let $0 \leq a_{kj} < \infty$ for all $1 \leq j \leq n$ and $1 \leq k \leq K$. Then*

$$\sum_{k=1}^{K} \prod_{j=1}^{n} a_{kj}^{\frac{1}{n}} \leq \prod_{j=1}^{n} \Big(\sum_{k=1}^{K} a_{kj} \Big)^{\frac{1}{n}}.$$

Proof. Use induction on the 2-product Hölder's inequality. □

Minkowski's inequality for finite sums can be extended to hold for countable sums.

Theorem 3.1.3. *(*Minkowski's inequality for countable sums*) Let $p \geq 1$. Let $f_j : \mathbb{R}^n \to \mathbb{C}$, $j \in \mathbb{N}_+$ be Lebesgue-measurable functions. Then*

$$\| \sum_{j=1}^{\infty} |f_j| \|_{L^p(\mathbb{R}^n)} \leq \sum_{j=1}^{\infty} \|f_j\|_{L^p(\mathbb{R}^n)}.$$

Proof.

$$\begin{aligned}
\| \sum_{j=1}^{\infty} |f_j| \|_{L^p(\mathbb{R}^n)} &= \Big(\int \Big(\sum_{j=1}^{\infty} |f_j(x)| \Big)^p \,\mathrm{d}x \Big)^{\frac{1}{p}} \\
&= \Big(\int \lim_{N \to \infty} \Big(\sum_{j=1}^{N} |f_j(x)| \Big)^p \,\mathrm{d}x \Big)^{\frac{1}{p}} \\
&= \Big(\lim_{N \to \infty} \int \Big(\sum_{j=1}^{N} |f_j(x)| \Big)^p \,\mathrm{d}x \Big)^{\frac{1}{p}} \qquad \text{(monotone convergence)} \\
&= \lim_{N \to \infty} \Big(\int \Big(\sum_{j=1}^{N} |f_j(x)| \Big)^p \,\mathrm{d}x \Big)^{\frac{1}{p}} \\
&\leq \lim_{N \to \infty} \sum_{j=1}^{N} \Big(\int |f_j(x)|^p \,\mathrm{d}x \Big)^{\frac{1}{p}} \qquad \text{(Minkowski)} \\
&= \sum_{j=1}^{\infty} \|f_j\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

□

Theorem 3.1.4. *(*Young's convolution inequality*) Let $f, g : \mathbb{R}^n \to \mathbb{C}$. Let $1 \leq p, q, r \leq \infty$ be such that $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$ (with interpretation $\frac{1}{\infty} = 0$). If $f \in L^p(\mathbb{R}^n)$ and $g \in L^r(\mathbb{R}^n)$, then $f * g \in L^q(\mathbb{R}^n)$ and*

$$\|f * g\|_q \leq \|f\|_p \|g\|_r.$$

Proof. We skip the proof, but this can be proven using Hölder's inequality and Fubini's theorem. A proof (albeit in a more general setting) can be found in [8], Theorem 1.2.12. □

Theorem 3.1.5. *(*The reverse Minkowski's inequality in l^p*) Let* $0 < p \leq 1$*. For* $u, v \in l^p$*, where*

$$l^p := \Big\{ z = (z_k)_{k=1}^{\infty} : z_k \in \mathbb{C} \text{ and } \sum_{k=1}^{\infty} |z_k|^p < \infty \Big\},$$

we have

$$\Big(\sum_{k=1}^{\infty} (|u_k| + |v_k|)^p \Big)^{\frac{1}{p}} \geq \Big(\sum_{k=1}^{\infty} |u_k|^p \Big)^{\frac{1}{p}} + \Big(\sum_{k=1}^{\infty} |v_k|^p \Big)^{\frac{1}{p}}.$$

Proof. We use the notation $\|u\|_p := \Big(\sum_{k=1}^{\infty} |u_k|^p \Big)^{\frac{1}{p}}$ and the same notation for v. (Notice though that if $p < 1$, this does not define a norm!) First of all,

$$(|u_k| + |v_k|)^p \leq 2^p (\max\{|u_k|, |v_k|\})^p \leq 2^p (|u_k|^p + |v_k|^p),$$

which implies that

$$\sum_{k=1}^{\infty} (|u_k| + |v_k|)^p \leq 2^p (\|u\|_p^p + \|v\|_p^p) < \infty.$$

We can assume that $\|u\|_p > 0$ and $\|v\|_p > 0$. We use the concavity of the function $x \mapsto x^p$, $x \geq 0$. For $t \in (0, 1)$,

$$(|u_k| + |v_k|)^p = \Big(t \frac{|u_k|}{t} + (1-t) \frac{|v_k|}{1-t} \Big)^p \geq t \frac{|u_k|^p}{t^p} + (1-t) \frac{|v_k|^p}{(1-t)^p}.$$

Taking the sum, we get for $t \in (0, 1)$

$$\sum_{k=1}^{\infty} (|u_k| + |v_k|)^p \geq \sum_{k=1}^{\infty} t \frac{|u_k|^p}{t^p} + \sum_{k=1}^{\infty} (1-t) \frac{|v_k|^p}{(1-t)^p}.$$

By choosing $t = \frac{\|u\|_p}{\|u\|_p + \|v\|_p}$ we get

$$\begin{aligned} \sum_{k=1}^{\infty} (|u_k| + |v_k|)^p &\geq t \sum_{k=1}^{\infty} \frac{|u_k|^p}{\frac{\|u\|_p^p}{(\|u\|_p + \|v\|_p)^p}} + (1-t) \sum_{k=1}^{\infty} \frac{|v_k|^p}{(1 - \frac{\|u\|_p}{\|u\|_p + \|v\|_p})^p} \\ &= t(\|u\|_p + \|v\|_p)^p + (1-t)(\|u\|_p + \|v\|_p)^p \\ &= (\|u\|_p + \|v\|_p)^p. \end{aligned}$$

□

Remark 3.1.6. Applying induction to Theorem 3.1.5, we can extend the claim to hold for all finite collections $\{u_1, \dots, u_m\}$ consisting of sequences in l^p:

$$\Big(\sum_{k=1}^{\infty}\Big(\sum_{j=1}^{m}|(u_j)_k|\Big)^p\Big)^{\frac{1}{p}} \geq \sum_{j=1}^{m}\Big(\sum_{k=1}^{\infty}|(u_j)_k|^p\Big)^{\frac{1}{p}}.$$

The following theorem is Theorem 7.26 from [12]. The proof is given there.

Theorem 3.1.7. *(*Change of variables using differentiability*) Suppose that*

(1) $X \subset V \subset \mathbb{R}^n$*,* V *is open,* $T : V \to \mathbb{R}^n$ *is continuous;*

(2) X *is measurable,* $T : X \to \mathbb{R}^n$ *is an injection, and* T *is differentiable at every point of* X*;*

(3) $m_n(T(V \setminus X)) = 0$.

Then

$$\int_{T(V)} f(x)\,\mathrm{d}x = \int_V f(T(x))|\det T'(x)|\,\mathrm{d}x,$$

where $f : \mathbb{R}^n \to [0, \infty)$ *is measurable or* $f : \mathbb{R}^n \to \mathbb{C}$ *is integrable.*

Remark 3.1.8. Also, it is noted in [12] that if $V \subset \mathbb{R}^n$ is open and $T : V \to \mathbb{R}^n$ is differentiable at every point of V, then T maps sets of measure 0 to sets of measure 0.

The following theorem is given without a proof. It can be found in Section 2.2 in [8].

Theorem 3.1.9. *(*Plancherel*) Let* $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$*. Then* $\hat{f} \in L^2(\mathbb{R}^n)$ *and*

$$\|\hat{f}\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}.$$

3.2 Schwartz functions

Schwartz functions will be used in this thesis in conjunction with the operator lemmas that are formulated in Section 4.2.

Definition 3.2.1. The class $\mathcal{S}(\mathbb{R}^n)$ of *Schwartz functions* is defined by

$$\mathcal{S}(\mathbb{R}^n) := \{f \in C^\infty(\mathbb{R}^n) \mid \|x^\beta \partial^\alpha f\|_\infty < \infty \ \forall \alpha, \beta \in \mathbb{N}^n\},$$

where

$$\|x^\beta \partial^\alpha f\|_\infty := \sup_{x \in \mathbb{R}^n} |x^\beta \partial^\alpha f(x)|$$

and

$$C^\infty(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{C} \mid \partial^\alpha f \text{ exists and is continuous for all } \alpha \in \mathbb{N}^n\}.$$

Remark 3.2.2. In Definition 3.2.1, we used notations $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$.

Schwartz functions decrease rapidly. Here is one (specific) manifestation of this phenomenon:

Lemma 3.2.3. *Let $\phi : \mathbb{R}^n \to [0, \infty)$ be a Schwartz function. Let $s > 0$. Then*

$$\phi(x) \lesssim_{\phi,s,n} \frac{1}{(1+|x|)^s}$$

for all $x \in \mathbb{R}^n$.

Proof. We divide the proof into two cases.

(1) Let $|x| \leq 1$. Then

$$(1+|x|)^s \phi(x) \leq 2^s \phi(x) \lesssim_s \phi(x) \leq \|\phi\|_{L^\infty} \lesssim_n 1.$$

(2) Let $|x| > 1$. Below, when $\beta \in \mathbb{N}^n$, we denote $|\beta| := \beta_1 + \cdots + \beta_n$. Using the multinomial theorem we get

$$\begin{aligned}
(1+|x|)^s \phi(x) \leq 2^s |x|^s \phi(x) &\lesssim_s |x|^s \phi(x) \\
&\leq (|x|^2)^{\lceil s \rceil} \phi(x) \\
&= \sum_{\substack{\beta \in \mathbb{N}^n \\ |\beta| = \lceil s \rceil}} \binom{\lceil s \rceil}{\beta} x^{2\beta} \phi(x) \\
&\leq \sum_{\substack{\beta \in \mathbb{N}^n \\ |\beta| = \lceil s \rceil}} \binom{\lceil s \rceil}{\beta} \|x^{2\beta} \phi\|_{L^\infty} \\
&\lesssim_{\phi,s,n} 1.
\end{aligned}$$

In the last line, we used the definition of a Schwartz function in order to deduce that the L^∞-norms are finite.

By combining (1) and (2) we get

$$\phi(x) \lesssim_{\phi,s,n} \frac{1}{(1+|x|)^s}.$$

□

Corollary 3.2.4. *Let $\phi : \mathbb{R}^n \to [0, \infty)$ be a Schwartz function. Let $p \geq 1$ and $s > 0$. Then*

$$\phi(x)^p \lesssim_{\phi,s,p,n} \frac{1}{(1+|x|)^s}.$$

for all $x \in \mathbb{R}^n$.

Proof. Substitute $s \mapsto s/p$ in Lemma 3.2.3 and raise both sides to power p. □

3.3 Fourier transforms of Schwartz functions

A consequence of Lemma 3.2.3 is that $\phi \lesssim_{\phi,n} w_{B(0,1),100n}$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$, which by Lemma 2.4.2 implies that ϕ is integrable on R^n. Since a Schwartz function is integrable on $\mathbb{R}^n$, its Fourier transform is defined. Let $\widetilde{f}$ denote the reflection of f, that is $\widetilde{f}(x) = f(-x)$ for all $x \in \mathbb{R}^n$. Let $*$ denote convolution and $\bar{f}$ be the complex conjugate of f.

If $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $a > 0$ and $y \in \mathbb{R}^n$, then the following hold:

- $\widetilde{f} \in \mathcal{S}(\mathbb{R}^n)$,
- $f * g \in \mathcal{S}(\mathbb{R}^n)$,
- $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$,
- $\widehat{f * g} = \hat{f}\hat{g}$, holds also for $f, g \in L^1(\mathbb{R}^n)$.
- $\widetilde{\hat{\hat{f}}} = f$,
- if $f \geq 0$, then $\widehat{\widetilde{f}} = \bar{\hat{f}}$,
- if $h(x) = f(ax)$ for $x \in \mathbb{R}^n$, then $h \in \mathcal{S}(\mathbb{R}^n)$ and $\hat{h}(x) = \frac{1}{a^n}\hat{f}(\frac{1}{a}x)$ for all $x \in \mathbb{R}^n$,
- if $h(x) = f(x - y)$ for $x \in \mathbb{R}^n$, then $h \in \mathcal{S}(\mathbb{R}^n)$ and $\hat{h}(x) = \mathrm{e}^{-\mathrm{i}2\pi y \cdot x}\hat{f}(x)$ for $x \in \mathbb{R}^n$.

We will prove one of the properties, but the other proofs of these properties are omitted. This topic is discussed e.g. in Section 2.2. in [8]. Let $f, g \in L^1(\mathbb{R}^n)$. Then by Fubini's theorem and a simple change of variables

$$\begin{aligned}
\widehat{f * g}(\xi) &= \int e(-\xi \cdot x) \int f(y) g(x - y) \, \mathrm{d}y \, \mathrm{d}x \\
&= \int f(y) \int g(x - y) e(-\xi \cdot x) \, \mathrm{d}x \, \mathrm{d}y \\
&= \int f(y) \int g(x) e(-\xi \cdot (x + y)) \, \mathrm{d}x \, \mathrm{d}y \\
&= \int f(y) e(-\xi \cdot y) \int g(x) e(-\xi \cdot x) \, \mathrm{d}x \, \mathrm{d}y \\
&= \int f(y) e(-\xi \cdot y) \hat{g}(\xi) \, \mathrm{d}y \\
&= \hat{f}(\xi)\hat{g}(\xi).
\end{aligned}$$

3.4 Some special Schwartz functions

Here we will prove the existence of particularly helpful Schwartz functions that we will use in proofs as auxiliary funtions.

Recall that given a function f, its *support*, $\mathrm{supp}(f)$, is the closure of the set $\{x \in \mathbb{R}^n \colon f(x) \neq 0\}$. A *compactly supported* function is a function that has a compact support.

Lemma 3.4.1. *A compactly supported function $f \in C^\infty(\mathbb{R}^n)$ is a Schwartz function.*

Proof. The claim follows from the definition of a Schwartz function, because each partial derivative of f is continuous and compactly supported. □

There are Schwartz functions whose graph resemble a "bump" of a given maximum diameter:

Lemma 3.4.2. *For each n, there exists a Schwartz function $f \in \mathcal{S}(\mathbb{R}^n)$, $f \geq 0$, such that*

$$\mathrm{supp}(f) \subset B(0, \frac{1}{2})$$

and

$$1 < \|f\|_{L^1(\mathbb{R}^n)} < \infty.$$

Proof. Define $\eta : \mathbb{R} \to [0, \infty)$,

$$\eta(t) := \begin{cases} \mathrm{e}^{\frac{1}{t^2-1}} & \text{if } |t| < 1, \\ 0 & \text{if } |t| \geq 1. \end{cases}$$

Observe that $\eta \in C^\infty(\mathbb{R})$ and $\|\eta\|_{L^1(\mathbb{R})} > 0$ (see e.g. [10], Exercise 2.8.6).

Define $g : \mathbb{R}^n \to [0, \infty)$ with

$$g(x) := \eta(4x_1) \cdots \eta(4x_n).$$

Then $g \in C^\infty(\mathbb{R}^n)$. Additionally, $\mathrm{supp}(g) \subset B(0, \frac{1}{2})$, since $g(x) \neq 0$ implies that $|x_k| < 1/4$ for all k.

Clearly $\|g\|_{L^1(\mathbb{R}^n)} > 0$. Let $a > 0$ be a real number such that $\|ag\|_{L^1(\mathbb{R}^n)} > 1$. Note that compactly supported C^∞-functions are Schwartz functions. Thus if we define $f := ag$, then f satisfies the required conditions. □

Furthermore, the "bump" can be flat, that is, the function can be constant in a given cube:

Lemma 3.4.3. *For each n, there exists a non-negative Schwartz function $\theta \in \mathcal{S}(\mathbb{R}^n)$ such that $\theta(x) = 1$ for all $x \in B(0, 1)$.*

Proof. Let g be as in the proof of Lemma 3.4.2. Define $h = bg$, where $b > 0$ is chosen so that $\|bg\|_{L^1(\mathbb{R}^n)} = 1$. Now $h \in C^\infty(\mathbb{R}^n)$, $\mathrm{supp}(h) \subset B(0, 1/2)$ and $h \geq 0$. Define

$$\theta = h * 1_{B(0,2)}.$$

Then $\theta \in C^\infty(\mathbb{R}^n)$ by Proposition 6.1.2 in [10]. Additionally, $\mathrm{supp}(\theta) \subset B(0, 4)$, which implies that θ is compactly supported. We have shown that θ is a Schwartz function.

Clearly $\theta \geq 0$. Fix $x \in B(0, 1)$. If $y \in \mathrm{supp}(h)$, then for all k it holds that $|x_k - y_k| \leq 1/2 + 1/4 < 1$. Hence $\mathrm{supp}(h) \subset B(x, 2)$. Thus

$$\theta(x) = \int_{\mathbb{R}^n} h(y) 1_{B(0,2)}(x - y)\,\mathrm{d}y = \int_{B(x,2)} h(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} h(y)\,\mathrm{d}y = 1.$$

□

Lemma 3.4.4. *For each n, there exists a non-negative Schwartz function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $1_{B(0,1)} \leq \varphi$ and $\mathrm{supp}(\hat{\varphi}) \subset B(0, 1)$.*

Proof. Let $f \in \mathcal{S}(\mathbb{R}^n)$ be as in Lemma 3.4.2. Denote $h := f * \widetilde{f}$. Then

$$\hat{h} = \hat{f}\hat{\tilde{f}} = \hat{f}\bar{\hat{f}} = |\hat{f}|^2,$$

since $f \geq 0$ (see section 3.3). Now

$$\hat{h}(0) = (\hat{f}(0))^2 = \|f\|^2_{L^1(\mathbb{R}^n)} > 1.$$

Since $\hat{h}$ is continuous, we find a $0 < \gamma < 1$ such that

$$\hat{h}(x) \geq 1,$$

when $x \in B(0, \gamma)$. Define

$$\phi := \tilde{\hat{h}} = \widetilde{|\hat{f}|^2}.$$

Then

$$\phi \geq 1_{B(0,\gamma)}$$

and

$$\phi \geq 0.$$

Also,

$$\hat{\phi} = h$$

and because $\mathrm{supp}(f) \subset B(0, 1/2)$ and $\mathrm{supp}(\tilde{f}) \subset B(0, 1/2)$, we get

$$\mathrm{supp}(\hat{\phi}) = \mathrm{supp}(f * \tilde{f}) \subset B(0, \frac{1}{2} + \frac{1}{2}) = B(0, 1).$$

Additionally, $\phi \in \mathcal{S}(\mathbb{R}^n)$.

Define $\varphi(x) := \phi(\gamma x)$ for all $x \in \mathbb{R}^n$. Then for $x \in \mathbb{R}^n$ it holds that

$$\varphi(x) \geq 1_{B(0,1)}(x).$$

Note that $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\varphi \geq 0$.

And if $\hat{\varphi}(x) = \frac{1}{\gamma^n}\hat{\phi}(\frac{1}{\gamma}x) \neq 0$, then $\frac{1}{\gamma}x \in B(0,1)$, which implies $x \in B(0,\gamma)$ and finally $x \in B(0,1)$. Hence $\operatorname{supp}(\hat{\varphi}) \subset B(0,1)$. $\square$

Chapter 4

Properties of the weight function

We begin to study covers of cubes with smaller cubes and the relations between the weight functions related to these cubes. We will also present the useful operator lemmas. The chapter ends with a reverse Hölder inequality.

Excluding the reverse Hölder inequality, the weight exponents E do not change in the proofs in this chapter. That is why it is justified to use the shorthand $w_B = w_{B,E}$ throughout.

4.1 Inequalities for weight functions

We start with a basic inequality that compares an indicator function to a weight function.

Lemma 4.1.1. *Let $E > n$ and $0 < R' \leq R$. Then*

$$1_B \lesssim_{E,n} \sum_{\Delta \in \mathcal{B}} w_{\Delta,E},$$

where $B = B(c_B, R) \subset \mathbb{R}^n$ is a cube and $\mathcal{B}$ is a cover of B with cubes of side length R'.

Proof. Let $x \in B$. Let $\Delta_x \in \mathcal{B}$ be centered at c_x be such that $x \in \Delta_x$. Now

$$w_{\Delta_x}(x) = \frac{1}{(1 + \frac{|x-c_x|}{R'})^E} \geq \frac{1}{(1 + \frac{\sqrt{n}R'}{2R'})^E} = \frac{1}{(1 + \frac{\sqrt{n}}{2})^E}.$$

Thus

$$1_B(x) = 1 \leq (1 + \frac{\sqrt{n}}{2})^E w_{\Delta_x}(x) \leq (1 + \frac{\sqrt{n}}{2})^E \sum_{\Delta \in \mathcal{B}} w_\Delta(x).$$

Let $x \in \mathbb{R}^n \setminus B$. Now $1_B(x) = 0 \leq (1 + \frac{\sqrt{n}}{2})^E \sum_{\Delta \in \mathcal{B}} w_\Delta(x)$.

Hence

$$1_B \lesssim_{E,n} \sum_{\Delta \in \mathcal{B}} w_\Delta.$$

□

By choosing $R' = R > 0$ and $\mathcal{B} = \{B\}$ in Lemma 4.1.1 we get the following result:

Corollary 4.1.2. *Let $E > n$. For all cubes $B = B(c, R) \subset \mathbb{R}^n$, where $R > 0$, we have*

$$1_B \lesssim_{E,n} w_{B,E}.$$

Actually, by doing a similar proof as in Lemma 4.1.1 we get the following corollary as well:

Corollary 4.1.3. *Let $E > n$. Let $B = B(c, R) \subset \mathbb{R}^n$ be a cube such that $R > 0$. Let $A = \{x \in \mathbb{R}^n \colon |x - c| \leq 2\sqrt{n}R\}$ be the ball centred at c. Then*

$$1_A \lesssim_{E,n} w_{B,E},$$

where the implicit constant is $(1 + 2\sqrt{n})^E$.

Next we present a result that is used to estimate the amount of essentially disjoint cubes of fixed size that fit inside a ball. This will be used in the upcoming lemmas. Recall that in an essentially disjoint collection of cubes the intersection of two members of the collection has measure zero.

Lemma 4.1.4. *Let $R > 0$. Let $\mathcal{B}$ be an essentially disjoint collection of cubes $B(c_B, R) \subset \mathbb{R}^n$. Let $k \in \mathbb{N}$. Then for each $x \in \mathbb{R}^n$ it holds that*

$$|\{\Delta \in \mathcal{B} \colon |x - c_\Delta| \leq 2^k R\}| \;\leq\; (2^{k+1} + \sqrt{n})^n.$$

Proof. Fix $x \in \mathbb{R}^n$. Assume that $\Delta \in \mathcal{B}$ is such that $|x - c_\Delta| \leq 2^k R$. Now for all $a \in \Delta$

$$|x - a| \leq |x - c_\Delta| + |c_\Delta - a| \leq 2^k R + \frac{\sqrt{n}R}{2} = \Big(2^k + \frac{\sqrt{n}}{2}\Big)R.$$

Thus $\Delta \subset \{a \in \mathbb{R}^n \colon |x - a| \leq (2^k + \frac{\sqrt{n}}{2})R\} \subset B(x, (2^{k+1} + \sqrt{n})R)$. In other words, Δ is contained in a cube centered at x with side length $(2^{k+1} + \sqrt{n})R$. (We used the fact that a ball in $\mathbb{R}^n$ with radius r is contained in the cube with side length $2r$ and the same center.)

Because the cubes of $\mathcal{B}$ are essentially disjoint (only their zero-measure boundaries may intersect) we can write

$$|\{\Delta \colon |x - c_\Delta| \leq 2^k R\}| \leq \frac{(2^{k+1} + \sqrt{n})^n R^n}{R^n} = (2^{k+1} + \sqrt{n})^n.$$

In the middle we have the quotient of the volumes of the container cube and a partition cube, that is, how many distinct partition cubes can fit inside $B(x,(2^{k+1}+\sqrt{n})R)$. $\square$

Next we pair Lemma 4.1.1 with another inequality. This time we restrict $\mathcal{B}$ to be a partition so that the overlap between the cubes is in control. The implicit constant in the new inequality is obtained with the help of a converging geometric series.

Lemma 4.1.5. *Fix $E > n$. Now*

$$1_B \lesssim_{E,n} \sum_{\Delta\in\mathcal{B}} w_{\Delta,E} \lesssim_{E,n} w_{B,E}$$

is valid for all cubes $B \subset \mathbb{R}^n$ with $l(B) = R$ and all essential partitions $\mathcal{B}$ of B with cubes Δ of fixed side length R', where $0 < R' \leq R$.

Proof. Let $0 < R' \leq R$. Let $B = B(c, R)$ and let $\mathcal{B}$ be an essential partition of B with cubes of side length R'.

By Lemma 4.1.1, it is enough to show the second inequality.

First, let us consider the case that $|x-c| \leq 2\sqrt{n}R$. This means that x is in B or relatively close to B. Now, by Corollary 4.1.3, it holds that $1 \lesssim_{E,n} w_B(x)$. To prove the claim, we show that the sum $\sum_\Delta w_\Delta$ is bounded from above by some constant that only depends on n and E.

We will divide the set $\mathcal{B}$ according to how far each center point c_Δ is from x. We use powers of 2 to categorise the distances.

Throughout this proof, inside set and sum notations, the symbol Δ is shorthand for $\Delta \in \mathcal{B}$. We can write

$$\begin{aligned}
\sum_\Delta w_\Delta(x) &= \sum_\Delta \Big(1 + \frac{|x-c_\Delta|}{R'}\Big)^{-E} \\
&= \sum_{\Delta\colon |x-c_\Delta|\leq R'} \Big(1 + \frac{|x-c_\Delta|}{R'}\Big)^{-E} + \sum_{k=1}^{\infty} \sum_{\substack{\Delta\colon \\ 2^{k-1}R' < |x-c_\Delta| \leq 2^k R'}} \Big(1 + \frac{|x-c_\Delta|}{R'}\Big)^{-E} \\
&\leq \sum_{\Delta\colon |x-c_\Delta|\leq R'} 1 \; + \; \sum_{k=1}^{\infty} \sum_{\substack{\Delta\colon \\ 2^{k-1}R' < |x-c_\Delta| \leq 2^k R'}} (2^{k-1})^{-E} \\
&\leq \sum_{\Delta\colon |x-c_\Delta|\leq R'} 1 \; + \; 2^E \cdot \sum_{k=1}^{\infty} \sum_{\Delta\colon |x-c_\Delta|\leq 2^k R'} 2^{-kE} \\
&= |\{\Delta\colon |x-c_\Delta| \leq R'\}| + 2^E \cdot \sum_{k=1}^{\infty} 2^{-kE} |\{\Delta\colon |x-c_\Delta| \leq 2^k R'\}| \qquad (4.1)
\end{aligned}$$

From the last row we see that we have reduced solving our upper bound to the question: How many center points c_Δ lie in each ball $A_k = \{a \in \mathbb{R}^n \colon |x-a| \le 2^k R'\}$, where $k = 0, 1, 2, \ldots$? Lemma 4.1.4 presents an upper bound for the amount of center points:

$$|\{\Delta \colon |x - c_\Delta| \le 2^k R'\}| \le (2^{k+1} + \sqrt{n})^n$$

for $k \in \mathbb{N}$.

Going back to our previous inequality, we can now write

$$\sum_\Delta w_\Delta(x) \le (2 + \sqrt{n})^n + 2^E \cdot \sum_{k=1}^{\infty} 2^{-kE}(2^{k+1} + \sqrt{n})^n.$$

Denote $M_k = \max\{2^{k+1}, 2^n\}$ for $k = 1, 2, \ldots$. Observe that $2^{k+1} + \sqrt{n} \le 2M_k$. We can write

$$\begin{aligned}
\sum_{k=1}^{\infty} 2^{-kE}(2^{k+1} + \sqrt{n})^n &= \sum_{k=1}^{\infty} 2^{-kE}(2M_k)^n \\
&= 2^n \sum_{k=1}^{\infty} 2^{-kE} M_k^n \\
&= 2^n \Big(\sum_{k=1}^{n-1} 2^{-kE} 2^{n^2} + \sum_{k=n}^{\infty} 2^{-kE} 2^{(k+1)n} \Big) \\
&= 2^n \Big(2^{n^2} \sum_{k=1}^{n-1} (2^{-E})^k + 2^n \sum_{k=n}^{\infty} (2^{n-E})^k \Big) \\
&:= U_{n,E},
\end{aligned} \tag{4.2}$$

where the series converges since $n - E < 0$. Observe that $0 < U_{n,E} < \infty$. Hence

$$\sum_\Delta w_\Delta(x) \le (2 + \sqrt{n})^n + 2^E \cdot U_{n,E} \lesssim_{E,n} 1 \lesssim_{E,n} w_B(x),$$

which finishes the proof in this first case.

Observe that we used the fact that the cubes do not overlap much ($\mathcal{B}$ is an essential partition) but we did **not** yet need the fact that all of the elements of $\mathcal{B}$ are close to B. It was enough that when x is in B or almost in B, then $w_B(x)$ can not be arbitrarily small, which in turn was ensured by the inequality $1 \lesssim_{E,n} w_B(x)$.

Next, let us consider the case $|x - c| > 2\sqrt{n}R$. We use the same expansion as in (4.1)

$$\sum_\Delta w_\Delta(x) \;\le\; |\{\Delta \colon |x - c_\Delta| \le R'\}| + 2^E \cdot \sum_{k=1}^{\infty} 2^{-kE} |\{\Delta \colon |x - c_\Delta| \le 2^k R'\}|.$$

We write $M := \frac{|x-c|}{R}$ and $N := \frac{\sqrt{n}}{2}$. Observe that

$$M - N > 2\sqrt{n} - \frac{\sqrt{n}}{2} = \frac{3}{2}\sqrt{n} \geq \frac{3}{2}. \tag{4.3}$$

Let $k \in \mathbb{N}$ be such that

$$\mathcal{A}_k := \{\Delta \in \mathcal{B} \colon |x - c_\Delta| \leq 2^k R'\} \neq \emptyset.$$

Then let $\Delta \in \mathcal{A}_k$. Now since $c_\Delta \in B(c, R)$,

$$|x - c| \leq |x - c_\Delta| + |c_\Delta - c| \leq 2^k R' + \frac{\sqrt{n}R}{2},$$

which implies that

$$2^k \geq \frac{|x-c| - \frac{\sqrt{n}R}{2}}{R'} = \frac{(M-N)R}{R'}. \tag{4.4}$$

We denote

$$k_0 := \min\Big\{k \in \mathbb{N}_+ \colon 2^k \geq \frac{(M-N)R}{R'}\Big\}.$$

Since $\mathcal{B}$ is an essential partition of $B(c,R)$, we have that $|\mathcal{B}| = (R/R')^n \in \mathbb{N}$. Using the aforementioned, we can write

$$\begin{aligned}
\sum_{k=1}^{\infty} 2^{-kE} |\{\Delta \colon |x - c_\Delta| \leq 2^k R'\}| &= \sum_{k=k_0}^{\infty} 2^{-kE} |\{\Delta \colon |x - c_\Delta| \leq 2^k R'\}| \\
&\leq \sum_{k=k_0}^{\infty} (2^{-E})^k \Big(\frac{R}{R'}\Big)^n \\
&= \Big(\frac{R}{R'}\Big)^n \sum_{k=0}^{\infty} (2^{-E})^{k+k_0} \\
&= 2^{-k_0 E} \Big(\frac{R}{R'}\Big)^n \sum_{k=0}^{\infty} (2^{-E})^k \\
&= (2^{k_0})^{-E} \Big(\frac{R}{R'}\Big)^n \frac{1}{1 - 2^{-E}} \\
&\leq \Big(\frac{(M-N)R}{R'}\Big)^{-E} \Big(\frac{R}{R'}\Big)^n \frac{1}{1 - 2^{-E}} \\
&\leq (M-N)^{-E} \frac{1}{1 - 2^{-E}} \\
&\lesssim_E (M-N)^{-E},
\end{aligned}$$

since $R/R' \geq 1$ and $n - E < 0$.

By our assumption $\frac{M}{2} > N$ which implies that $M = 2(M - \frac{M}{2}) < 2(M - N)$. Thus $M \lesssim M - N$.

Additionally, since $M \geq 1/2$, we can write

$$M + 1 \leq M + 2M = 3M,$$

whence $M + 1 \lesssim M$.

Combining these we can write $(M-N)^{-E} \lesssim_E (M+1)^{-E}$ and thus

$$\sum_{k=1}^{\infty} 2^{-kE} |\{\Delta \colon |x - c_\Delta| \leq 2^k R'\}| \lesssim_E (M+1)^{-E}$$

We will then show that $\mathcal{A}_0 = \emptyset$. If on the contrary $\mathcal{A}_0$ was non-empty, then as seen earlier, (4.4) would hold with $k = 0$. Combining this with (4.3) we would get

$$1 \geq \frac{(M-N)R}{R'} > \frac{3}{2} > 1,$$

which is a contradiction. Thus $\mathcal{A}_0 = \emptyset$.

Thus

$$\sum_{\Delta} w_\Delta(x) \lesssim_E 2^E \cdot (M+1)^{-E} \lesssim_E (M+1)^{-E} = w_B(x),$$

which finishes the proof in the other case.

After taking the maximum of the implicit constants of the two cases we have proved the claim. $\square$

Lemma 4.1.6. *Let $E > n$. Let $0 < R' \leq R$ and let $C > 0$. If $B = B(c, R)$ and $B' = B'(c', R')$ are cubes in $\mathbb{R}^n$ and $|c - c'| \leq CR$, then*

$$w_{B',E}(x) \lesssim_{C,E} w_{B,E}(x)$$

for all $x \in \mathbb{R}^n$.

Proof. Assume first that $|x - c| \leq 2CR$. Then

$$w_{B',E}(x) = \Big(1 + \frac{|x-c'|}{R'}\Big)^{-E} \leq 1 \lesssim_{C,E} \Big(1 + 2C\Big)^{-E} \leq \Big(1 + \frac{|x-c|}{R}\Big)^{-E} = w_{B,E}(x).$$

Assume for the rest of the proof that $|x-c| \geq 2CR$. Then

$$\begin{aligned}
\Big(1+\frac{|x-c'|}{R'}\Big)^{-E} &\leq \Big(1+\frac{|x-c'|}{R}\Big)^{-E} \\
&\leq \Big(1+\frac{|x-c|}{R}-\frac{|c-c'|}{R}\Big)^{-E} \\
&\leq \Big(1+\frac{|x-c|}{R}-\frac{CR}{R}\Big)^{-E} \\
&\leq \Big(1+\frac{|x-c|}{R}-C\Big)^{-E} \\
&\leq \Big(1+\frac{|x-c|}{R}-\frac{|x-c|}{2R}\Big)^{-E} \\
&= \Big(1+\frac{|x-c|}{2R}\Big)^{-E} \\
&\lesssim_E \Big(1+\frac{|x-c|}{R}\Big)^{-E}.
\end{aligned}$$

□

4.2 Operators related to weight functions

Before delving into the operators, we start with a lemma.

Lemma 4.2.1. *Fix $E > n$. Let $R > 0$ and let $B = B(c_B, R)$ be a cube in $\mathbb{R}^n$. Let $\mathcal{B}$ be an essential partition of $\mathbb{R}^n$ with cubes $B' = B'(c_{B'}, R)$. Then*

$$w_{B,E}(x) \lesssim_{E,n} \sum_{B' \in \mathcal{B}} 1_{B'}(x) w_{B,E}(c_{B'}) \tag{4.5}$$

and

$$\sum_{B' \in \mathcal{B}} w_{B',E}(x) w_{B,E}(c_{B'}) \lesssim_{E,n} w_{B,E}(x) \tag{4.6}$$

for all $x \in \mathbb{R}^n$.

Proof. Let $R > 0$. Let B be a cube in $\mathbb{R}^n$ with side length R. Denote the center of B with c_B.

Let $\mathcal{B}$ be an essential partition of $\mathbb{R}^n$ with cubes $B' = B'(c_{B'}, R)$. Since $\mathbb{R}^n$ is separable, the partition $\mathcal{B}$ is countable.

Proof of (4.5): Let $x \in \mathbb{R}^n$. Let $B'_x \in \mathcal{B}$ such that $x \in B'_x$. Now

$$\begin{aligned}\frac{|c_{B'_x} - c_B|}{R} &\leq \frac{|x - c_{B'_x}| + |x - c_B|}{R} \\ &\leq \frac{\frac{\sqrt{n}R}{2} + |x - c_B|}{R} \\ &= \frac{\sqrt{n}}{2} + \frac{|x - c_B|}{R}\end{aligned}$$

and thus

$$\begin{aligned}\frac{1 + \frac{|c_{B'_x} - c_B|}{R}}{1 + \frac{|x - c_B|}{R}} &\leq \frac{1 + \frac{\sqrt{n}}{2} + \frac{|x - c_B|}{R}}{1 + \frac{|x - c_B|}{R}} \\ &\leq 1 + \frac{\sqrt{n}}{2}.\end{aligned}$$

Hence

$$\begin{aligned}w_B(x) &= \frac{1}{(1 + \frac{|x - c_B|}{R})^E} \\ &\leq \frac{(1 + \frac{\sqrt{n}}{2})^E}{(1 + \frac{|c_{B'_x} - c_B|}{R})^E} \\ &= (1 + \frac{\sqrt{n}}{2})^E w_B(c_{B'_x}) \\ &\leq (1 + \frac{\sqrt{n}}{2})^E \sum_{B' \in \mathcal{B}} 1_{B'}(x) w_B(c_{B'})\end{aligned}$$

and thus

$$w_B(x) \lesssim_{E,n} \sum_{B' \in \mathcal{B}} 1_{B'}(x) w_B(c_{B'}).$$

Proof of (4.6): Because $\mathcal{B}$ is an essential partition, by Lemma 4.1.4 it holds for all $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$ that

$$|\{B' \in \mathcal{B} \colon |x - c_{B'}| \leq 2^k R\}| \ \leq \ (2^{k+1} + \sqrt{n})^n.$$

This estimate will be used freely throughout the proof and it also implies that for all $x \in \mathbb{R}^n$

$$\begin{aligned}
\sum_{B'\colon |x-c_{B'}|>R} w_{B'}(x) &\le \sum_{k=1}^{\infty} \sum_{\substack{B'\colon \\ 2^{k-1}R<|x-c_{B'}|\le 2^k R}} w_{B'}(x) \\
&\le \sum_{k=1}^{\infty} \sum_{\substack{B'\colon \\ 2^{k-1}R<|x-c_{B'}|\le 2^k R}} 2^{-kE}2^E \\
&\lesssim_E \sum_{k=1}^{\infty} 2^{-kE}|\{B'\colon |x-c_{B'}| \le 2^k R\}| \\
&\le \sum_{k=1}^{\infty} 2^{-kE}(2^{k+1}+\sqrt{n})^n \\
&\lesssim_{E,n} 1,
\end{aligned}$$

where the last inequality holds, because $E > n$ (see (4.2)). We deduced that

$$\sum_{B'\colon |x-c_{B'}|>R} w_{B'}(x) \lesssim_{E,n} 1 \tag{4.7}$$

for all $x \in \mathbb{R}^n$.

(1) First let $x \in \mathbb{R}^n$ be such that $|x - c_B| \le 2\sqrt{n}R$. Now, by Corollary 4.1.3 it holds that $1 \lesssim_{E,n} w_B(x)$. Then by (4.7)

$$\begin{aligned}
\sum_{B'\in\mathcal{B}} w_{B'}(x)w_B(c_{B'}) &\le \sum_{B'\colon |x-c_{B'}|\le R} 1 \;+\; \sum_{B'\colon |x-c_{B'}|>R} w_{B'}(x) \\
&\lesssim_{E,n} |\{B'\colon |x-c_{B'}| \le R\}| \;+\; 1 \\
&\le (2+\sqrt{n})^n + 1 \\
&\lesssim_n 1 \\
&\lesssim_{E,n} w_B(x),
\end{aligned}$$

This case was similar to the proof of the respective case in Lemma 4.1.5.

(2) Then let $x \in \mathbb{R}^n$ be such that $|x - c_B| > 2\sqrt{n}R$. We denote

$\mathcal{H} := \{B'\colon |x - c_{B'}| \le R\}$,

$\mathcal{I} := \{B'\colon |c_B - c_{B'}| \le R\}$,

$\mathcal{J} := \{B'\colon \min\{|x - c_{B'}|, |c_B - c_{B'}|\} > R\}$,

$\mathcal{S} := \mathcal{J} \cap \{B'\colon |x - c_{B'}| \ge |c_B - c_{B'}|\}$,

$\mathcal{T} := \mathcal{J} \cap \{B'\colon |x - c_{B'}| < |c_B - c_{B'}|\}$.

Observe that if $B' \in \mathcal{H}$, then $|c_B - c_{B'}| \geq |x - c_B| - |x - c_{B'}| \geq |x - c_B| - R > (2\sqrt{n} - 1)R \geq R$. Similarly, if $B' \in \mathcal{I}$, then $|x - c_{B'}| \geq |x - c_B| - |c_B - c_{B'}| \geq |x - c_B| - R > (2\sqrt{n} - 1)R \geq R$. Hence it holds that $\mathcal{H} \cap \mathcal{I} = \emptyset$ and thus $\mathcal{B} = \mathcal{H} \cup \mathcal{I} \cup \mathcal{S} \cup \mathcal{T}$, where the union is disjoint. Hence we can write

$$\sum_{B' \in \mathcal{B}} w_{B'}(x) w_B(c_{B'}) = \Big(\sum_{B' \in \mathcal{H}} + \sum_{B' \in \mathcal{I}} + \sum_{B' \in \mathcal{S}} + \sum_{B' \in \mathcal{T}} \Big) w_{B'}(x) w_B(c_{B'}),$$

although in this case a non-disjoint union and therefore an inequality would suffice. We note that it is enough to bound each of these four sums separately with $w_B(x)$ multiplied by a constant that depends only on E and n.

If $B' \in \mathcal{H}$, then

$$\begin{aligned} w_B(c_{B'}) \leq \Big(1 + \frac{|x - c_B| - R}{R}\Big)^{-E} = \Big(\frac{|x - c_B|}{R}\Big)^{-E} &\lesssim_E \Big(2 \cdot \frac{|x - c_B|}{R}\Big)^{-E} \\ &\leq w_B(x), \end{aligned}$$

where our assumption $|x - c_B| > R$ guarantees the last inequality.

If $B' \in \mathcal{I}$, then similarly

$$w_{B'}(x) \leq \Big(1 + \frac{|x - c_B| - R}{R}\Big)^{-E} \lesssim_E w_B(x).$$

We get

$$\begin{aligned} \sum_{B' \in \mathcal{H}} w_{B'}(x) \underbrace{w_B(c_{B'})}_{\lesssim_E w_B(x)} \lesssim_E w_B(x) \cdot \sum_{B' \in \mathcal{H}} w_{B'}(x) \leq w_B(x) \cdot \sum_{B' \in \mathcal{H}} 1 & \\ \leq |\{B' \colon |x - c_{B'}| \leq R\}| \cdot w_B(x) & \\ \leq (2 + \sqrt{n})^n w_B(x) & \end{aligned}$$

and similarly

$$\sum_{B' \in \mathcal{I}} \underbrace{w_{B'}(x)}_{\lesssim_E w_B(x)} \underbrace{w_B(c_{B'})}_{\leq 1} \lesssim_E |\{B' \colon |c_B - c_{B'}| \leq R\}| \cdot w_B(x) \leq (2 + \sqrt{n})^n w_B(x).$$

Denote $M := |x - c_B|/R$. Then

$$1 + M \lesssim \frac{1}{2} + \frac{M}{2} \leq 1 + \frac{M}{2}.$$

Let $B' \in \mathcal{S}$. Then

$$\frac{|x - c_B|}{2} \leq \frac{|x - c_{B'}| + |c_B - c_{B'}|}{2} \leq \frac{2 \cdot |x - c_{B'}|}{2} = |x - c_{B'}|,$$

and thus
$$w_{B'}(x) \leq (1+\frac{M}{2})^{-E} \lesssim_E (1+M)^{-E} = w_B(x).$$
Hence
$$\sum_{B' \in \mathcal{S}} w_{B'}(x) w_B(c_{B'}) \lesssim_E w_B(x) \cdot \sum_{B' \in \mathcal{S}} w_B(c_{B'}),$$
where by (4.7)
$$\sum_{B' \in \mathcal{S}} w_B(c_{B'}) \leq \sum_{B' \colon |c_B - c_{B'}| > R} w_B(c_{B'}) = \sum_{B' \colon |c_B - c_{B'}| > R} w_{B'}(c_B) \lesssim_{E,n} 1.$$

Let $B' \in \mathcal{T}$. Then (similarly as in the case where $\mathcal{S}$ is considered)
$$\frac{|x - c_B|}{2} \leq \frac{|x - c_{B'}| + |c_B - c_{B'}|}{2} \leq \frac{2 \cdot |c_B - c_{B'}|}{2} = |c_B - c_{B'}|,$$
and thus
$$w_B(c_{B'}) \leq (1+\frac{M}{2})^{-E} \lesssim_E (1+M)^{-E} = w_B(x).$$
Hence
$$\sum_{B' \in \mathcal{T}} w_{B'}(x) \underbrace{w_B(c_{B'})}_{\lesssim_E w_B(x)} \lesssim_E w_B(x) \cdot \sum_{B' \in \mathcal{T}} w_{B'}(x),$$
where by (4.7)
$$\sum_{B' \in \mathcal{T}} w_{B'}(x) \leq \sum_{B' \colon |x - c_{B'}| > R} w_{B'}(x) \lesssim_{E,n} 1.$$

We have proved that for all $x \in \mathbb{R}^n$
$$\sum_{B' \in \mathcal{B}} w_{B'}(x) w_B(c_{B'}) \lesssim_{E,n} w_B(x).$$

□

Next we consider $L^1_+(\mathbb{R}^n)$, the space of non-negative functions in $L^1(\mathbb{R}^n)$. For us, these functions act as weight functions, so we use the letter $\mathcal{W}$ to denote the space $L^1_+(\mathbb{R}^n)$.

We consider operators that act on $\mathcal{W}$ and that are of a specific type. We derive a way to determine inequalities between the values of these operators. Then we construct an example of such operators. These kinds of operators will arise in the proofs of 4.4.3, 5.1.1 and 5.2.3 and it is beneficial not to repeat the same arguments in these proofs, as made possible by the following lemmas.

Lemma 4.2.2. *Fix $E > n$. Fix $R > 0$. Let the operators $O_1, O_2 : \mathcal{W} \to [0, \infty]$ have the following four properties:*

(W1) $O_1(1_B) \lesssim O_2(w_{B,E})$ *for all cubes* $B \subset \mathbb{R}^n$ *of side length* R*, where the implicit constant is independent of the center point of* B *but is allowed to depend on* R*,* E *and* n*.*

(W2) $O_1(\sum_{k=1}^{\infty} \alpha_k u_k) \leq \sum_{k=1}^{\infty} \alpha_k O_1(u_k)$ *for all* $u_k \in \mathcal{W}$ *and* $\alpha_k \in (0,\infty)$ *such that* $\sum_{k=1}^{\infty} \alpha_k u_k \in \mathcal{W}$*.*

(W3) $O_2(\sum_{k=1}^{\infty} \alpha_k u_k) \geq \sum_{k=1}^{\infty} \alpha_k O_2(u_k)$ *for all* $u_k \in \mathcal{W}$ *and* $\alpha_k \in (0,\infty)$ *such that* $\sum_{k=1}^{\infty} \alpha_k u_k \in \mathcal{W}$*.*

(W4) If $u \leq v$*,* $u, v \in \mathcal{W}$*, then* $O_j(u) \leq O_j(v)$*, where* $j \in \{1,2\}$*.*

Then

$$O_1(w_{B,E}) \lesssim O_2(w_{B,E}) \tag{4.8}$$

for each cube $B \subset \mathbb{R}^n$ *with side length* R*. The implicit constant in* (4.8) *only depends on the implicit constant from (W1), on* E *and* n*.*

Proof. Fix $E > n$. Fix $R > 0$. Denote the implicit constant in (W1) by $C_1 > 0$.

Let $z : \mathbb{R}^n \to \mathbb{R}$ be the zero function i.e. $z(x) = 0$ for all $x \in \mathbb{R}^n$.

Assume first that $O_1(z) = \infty$. Let B be a cube in $\mathbb{R}^n$ with side length R. Now by (W4) it holds that $O_1(1_B) = \infty$. Thus by (W1) we can write $O_2(w_{B,E}) = \infty$, which implies that $O_1(w_{B,E}) \lesssim O_2(w_{B,E})$, where the implicit constant can be, for example, chosen to be 1.

Assume then that $O_2(z) = \infty$. Let B be a cube in $\mathbb{R}^n$ with side length R. Now by (W4) $O_2(w_{B,E}) = \infty$ and this implies that $O_1(w_{B,E}) \lesssim O_2(w_{B,E})$, where the implicit constant can be, for example, chosen to be 1.

Assume for the rest of the proof that $O_1(z) < \infty$ and $O_2(z) < \infty$. By (W2) we can write

$$O_1(z) = O_1(\sum_{k=1}^{\infty} \frac{1}{2^{k+1}} z) \leq \frac{1}{2} O_1(z),$$

which implies that $O_1(z) = 0$. Additionally, by (W3) we can write

$$2\, O_2(z) \leq O_2(\sum_{k=1}^{\infty} \frac{1}{2^{k-1}} z) = O_2(z),$$

which implies that $O_2(z) = 0$. Observe that using the fact that $O_1(z) = O_2(z) = 0$ we get that conditions (W2)-(W3) hold also for finite sums. In particular, with $u_2, u_3, \ldots = z$ we get

$$O_1(\alpha u) \leq \alpha O_1(u) = \alpha O_1(\alpha^{-1} \alpha u) \leq O_1(\alpha u)$$

for all $u \in \mathcal{W}$ and $\alpha \in (0, \infty)$. We can do a similar deduction for O_2 and hence we get

$$O_j(\alpha u) = \alpha O_j(u) \tag{4.9}$$

for $j \in \{1, 2\}$, $u \in \mathcal{W}$ and $\alpha \in (0, \infty)$.

Then let us prove (4.8). Let B be a cube in $\mathbb{R}^n$ with side length R. Let $\mathcal{B}$ be an essential partition of $\mathbb{R}^n$ with cubes $B' = B'(c_{B'}, R)$. Recall that $\mathcal{B}$ is *countable* because of the separability of $\mathbb{R}^n$.

By Lemma 4.2.1 for all $x \in \mathbb{R}^n$

$$w_B(x) \le C_2 \sum_{B' \in \mathcal{B}} 1_{B'}(x) w_B(c_{B'})$$

and

$$\sum_{B' \in \mathcal{B}} w_{B'}(x) w_B(c_{B'}) \le C_3 w_B(x),$$

where $C_2 > 0$ and $C_3 > 0$ only depend on E and n. With the help of Lemma 4.1.5 we have for all $x \in \mathbb{R}^n$ that

$$0 \le \sum_{B' \in \mathcal{B}} 1_{B'}(x) w_B(c_{B'}) \lesssim_{E,n} \sum_{B' \in \mathcal{B}} w_{B'}(x) w_B(c_{B'}) \le C_3 w_B(x), \tag{4.10}$$

and thus all of the functions appearing in (4.10) belong to $\mathcal{W}$, since $w_B \in \mathcal{W}$. This observation explains why all usages of the rules (W1)-(W4) and (4.9) below are valid.

$$\begin{aligned}
O_1(w_B) &\le O_1(C_2 \sum_{B' \in \mathcal{B}} w_B(c_{B'}) 1_{B'}) && \text{(W4)} \\
&= C_2\, O_1(\sum_{B' \in \mathcal{B}} w_B(c_{B'}) 1_{B'}) && \text{(4.9)} \\
&\le C_2 \sum_{B' \in \mathcal{B}} w_B(c_{B'}) O_1(1_{B'}) && \text{(W2)} \\
&\le C_2 \sum_{B' \in \mathcal{B}} w_B(c_{B'}) C_1\, O_2(w_{B'}) && \text{(W1)} \\
&= C_1 C_2 \sum_{B' \in \mathcal{B}} w_B(c_{B'}) O_2(w_{B'}) && \\
&\le C_1 C_2\, O_2(\sum_{B' \in \mathcal{B}} w_B(c_{B'}) w_{B'}) && \text{(W3)} \\
&\le C_1 C_2\, O_2(C_3\, w_B) && \text{(W4)} \\
&= C_1 C_2 C_3\, O_2(w_B) && \text{(4.9)}.
\end{aligned}$$

$\square$

Let $\eta : \mathbb{R}^n \to [0, \infty)$ be a function and $B = B(c, R)$ be a cube in $\mathbb{R}^n$. Throughout, we will consider the translated and scaled version $\eta_B : \mathbb{R}^n \to [0, \infty)$ defined by the formula $\eta_B(x) = \eta(\frac{x-c}{R})$.

We also need a version of the previous lemma, where instead of $O_2(w_B)$ in (W1) we have $O_2(\eta_B)$ for a certain rapidly decreasing function η.

Lemma 4.2.3. *Let φ be the function constructed in Lemma 3.4.4 and let $p \geq 1$. Denote $\eta = \varphi^p$. Fix $E > n$. Fix $R > 0$. Let the operators $O_1, O_2 : \mathcal{W} \to [0, \infty]$ have the following four properties:*

- *(V1) $O_1(1_B) \lesssim O_2(\eta_B)$ for all cubes $B \subset \mathbb{R}^n$ of side length R, where the implicit constant is independent of the center point of B but is allowed to depend on R, E, p and n.*
- *(V2) $O_1(\sum_{k=1}^{\infty} \alpha_k u_k) \leq \sum_{k=1}^{\infty} \alpha_k O_1(u_k)$ for all $u_k \in \mathcal{W}$ and $\alpha_k \in (0, \infty)$ such that $\sum_{k=1}^{\infty} \alpha_k u_k \in \mathcal{W}$.*
- *(V3) $O_2(\sum_{k=1}^{\infty} \alpha_k u_k) \geq \sum_{k=1}^{\infty} \alpha_k O_2(u_k)$ for all $u_k \in \mathcal{W}$ and $\alpha_k \in (0, \infty)$ such that $\sum_{k=1}^{\infty} \alpha_k u_k \in \mathcal{W}$.*
- *(V4) If $u \leq v$, $u, v \in \mathcal{W}$, then $O_j(u) \leq O_j(v)$, where $j \in \{1, 2\}$.*

Then

$$O_1(w_{B,E}) \lesssim O_2(w_{B,E}) \tag{4.11}$$

for each cube $B \subset \mathbb{R}^n$ with side length R. The implicit constant in (4.11) *only depends on the implicit constant from (V1), on E, p and n.*

Proof. Fix $E > n$. Fix $R > 0$. Denote the implicit constant in (V1) by $D_1 > 0$.

We start the proof with an observation. Let $B = B(c, R)$ be a cube. Because φ is a non-negative Schwartz function, then $\eta \in \mathcal{W}$. By Corollary 3.2.4

$$\eta \lesssim_{E,p,n} w_{B(0,1),E}.$$

Thus for all $x \in \mathbb{R}^n$ we have

$$\eta_B(x) = \eta\Big(\frac{x-c}{R}\Big) \lesssim_{E,p,n} w_{B(0,1),E}\Big(\frac{x-c}{R}\Big) = w_{B,E}(x). \tag{4.12}$$

Let us denote the implicit constant in (4.12) by $D = D_{E,p,n}$.

Then let us start the main part of the proof. Let $z : \mathbb{R}^n \to \mathbb{R}$ be the zero function i.e. $z(x) = 0$ for all $x \in \mathbb{R}^n$.

Assume first that $O_2(z) = \infty$. Let B be a cube in $\mathbb{R}^n$ with side length R. Now by (V4) it holds that $O_2(w_{B,E}) = \infty$ and this implies that $O_1(w_{B,E}) \lesssim O_2(w_{B,E})$, where the implicit constant can be, for example, chosen to be 1.

Assume then that $O_2(z) < \infty$. Let B be a cube in $\mathbb{R}^n$ with side length R. As in the proof of Lemma 4.2.2 we get

$$O_2(\alpha u) = \alpha O_2(u) \tag{4.13}$$

for $u \in \mathcal{W}$ and $\alpha \in (0, \infty)$. Then we aim to prove (W1). Let B be a cube in $\mathbb{R}^n$ with side length R. By (V1), (V4) and (4.13)

$$O_1(1_B) \leq D_1 O_2(\eta_B) \leq D_1 O_2(D w_{B,E}) = D_1 D O_2(w_{B,E}).$$

The constant $D_1 D$ is independent of the center point of B. Hence (W1) and conditions (W2)-(W4) hold. Thus Lemma 4.2.2 implies that

$$O_1(w_{B,E}) \lesssim O_2(w_{B,E})$$

holds for each cube $B \subset \mathbb{R}^n$ with side length R, where the implicit constant only depends on $D_1 D$, E and n. Since D is entirely determined by E, p and n, the claim follows. $\square$

Remark 4.2.4. Although in (W1) and (V1) the implicit constant is allowed to depend on R, we will mostly use these theorems in situations where it does not depend on R. This is because we often do not want the implicit constants in (4.8) and (4.11) to depend on R.

4.3 An example of operators that satisfy the conditions (W2)-(W4) of Lemma 4.2.2

We will construct a family of operators that satisfy conditions (W2)-(W4). Such operators will appear in proofs later on.

Here $n \geq 1$. Let $\mathcal{W}$ be the collection of all non-negative, integrable functions on $\mathbb{R}^n$.

First, let us fix a measurable function $f : \mathbb{R}^n \to \mathbb{C}$ and $p \geq 1$. We define $O_1 : \mathcal{W} \to [0, \infty]$ with the following formula:

$$O_1(v) := \int_{\mathbb{R}^n} |f(x)|^p v(x) \,\mathrm{d}x = \|f\|^p_{L^p(v)}$$

for all $v \in \mathcal{W}$.

We prove that O_1 satisfies the conditions (W2) and (W4) of Lemma 4.2.2.

Let $u_k \in \mathcal{W}$ and $\alpha_k \in (0, \infty)$ such that $w := \sum_{k=1}^{\infty} \alpha_k u_k \in \mathcal{W}$. Now

$$\begin{aligned} O_1(\sum_{k=1}^{\infty} \alpha_k u_k) &= \int_{\mathbb{R}^n} |f|^p \sum_{k=1}^{\infty} \alpha_k u_k \\ &= \sum_{k=1}^{\infty} \alpha_k \int_{\mathbb{R}^n} |f|^p u_k \\ &= \sum_{k=1}^{\infty} \alpha_k O_1(u_k), \end{aligned}$$

so (W2) holds. Assume then that $u \leq v$. Then

$$O_1(u) = \int_{\mathbb{R}^n} |f|^p u \leq \int_{\mathbb{R}^n} |f|^p v = O_1(v),$$

and thus (W4) holds.

Then fix $p \geq 2$ and measurable L^∞-functions $f_i : \mathbb{R}^n \to \mathbb{C}$, $i \in \mathbb{N}_+$. Define $O_2 : \mathcal{W} \to [0, \infty]$ with the following formula:

$$O_2(v) := (\sum_i \|f_i\|_{L^p(v)}^2)^{p/2}$$

for $v \in \mathcal{W}$. Choosing L^∞-functions guarantees that $\int_{\mathbb{R}^n} |f_i|^p v < \infty$ for all $i \in \mathbb{N}_+$ and $v \in \mathcal{W}$.

We prove that O_2 satisfies the conditions (W3) and (W4) of Lemma 4.2.2.

Assume that $u, v \in \mathcal{W}$, $u \leq v$. Then

$$O_2(u) = \Big(\sum_i \Big(\int_{\mathbb{R}^n} |f_i|^p u\Big)^{2/p}\Big)^{p/2} \leq \Big(\sum_i \Big(\int_{\mathbb{R}^n} |f_i|^p v\Big)^{2/p}\Big)^{p/2} = O_2(v),$$

and hence (W4) holds.

Next, let $u_k \in \mathcal{W}$ and $\alpha_k \in (0, \infty)$ such that $w := \sum_{k=1}^{\infty} \alpha_k u_k \in \mathcal{W}$. We can assume that $\sum_i \|f_i\|_{L^p(w)}^2 < \infty$. Now using monotone convergence and the reverse

Minkowski's inequality in $l_{2/p}$ we get

$$\begin{aligned}
\infty > O_2(w) &= \Big(\sum_i \Big(\int_{\mathbb{R}^n} |f_i|^p \sum_{k=1}^{\infty} \alpha_k u_k\Big)^{2/p}\Big)^{p/2} \\
&= \Big(\sum_i \Big(\sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |f_i|^p \alpha_k u_k\Big)^{2/p}\Big)^{p/2} \\
&= \Big(\sum_i \lim_{N\to\infty} \Big(\sum_{k=1}^{N} \int_{\mathbb{R}^n} |f_i|^p \alpha_k u_k\Big)^{2/p}\Big)^{p/2} \\
&\overset{\text{m. c.}}{=} \lim_{N\to\infty} \Big(\sum_i \Big(\sum_{k=1}^{N} \int_{\mathbb{R}^n} |f_i|^p \alpha_k u_k\Big)^{2/p}\Big)^{p/2} \\
&\overset{\text{r.M.}}{\geq} \lim_{N\to\infty} \sum_{k=1}^{N} \Big(\sum_i \Big(\int_{\mathbb{R}^n} |f_i|^p \alpha_k u_k\Big)^{2/p}\Big)^{p/2} \\
&= \sum_{k=1}^{\infty} \alpha_k \Big(\sum_i \Big(\int_{\mathbb{R}^n} |f_i|^p u_k\Big)^{2/p}\Big)^{p/2} \\
&= \sum_{k=1}^{\infty} \alpha_k O_2(u_k).
\end{aligned}$$

Hence (W3) holds.

We proved the following:

Proposition 4.3.1. *Let us fix $p \geq 2$ and functions f_i and f as above. Then the aforementioned operators O_1 and O_2 satisfy conditions (W2)-(W4) of Lemma 4.2.2.*

Remark 4.3.2. For certain f_i and f, such as the ones used in 4.4.3, 5.1.1 and 5.2.3, the condition (W1) or (V1) will hold as well. That will allow us to use the operator lemmas 4.2.2 and 4.2.3.

4.4 A reverse Hölder inequality

- Let A be a set. If $A \subset \mathbb{R}^n$ and A has positive Lebesgue measure, then $|A|$ will refer to the Lebesgue measure of A. Otherwise, $|A|$ will refer to the cardinality of A.
- We denote $\|F\|_{L^p_{\#}(w_{B,E})} := \frac{1}{|B|^{1/p}} \|F\|_{L^p(w_{B,E})}$, for $F \in L^p(w_{B,E})$.

Before stating the reverse Hölder inequality, we present the following lemmas.

Lemma 4.4.1. *Let $f, g \in L^1(\mathbb{R}^n)$. Then $f\tilde{\hat{g}} \in L^1(\mathbb{R}^n)$ and*

$$\widehat{f\tilde{\hat{g}}} = \hat{f} * g.$$

Proof. The first statement is proven by

$$\int |f(x)\tilde{g}(x)|\,\mathrm{d}x \le \int |f(x)|\Big|\int g(y)e(y\cdot x)\,\mathrm{d}y\Big|\,\mathrm{d}x \le \int |f(x)| \int |g(y)|\,\mathrm{d}y\,\mathrm{d}x$$
$$= \|f\|_{L^1(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)}.$$

Hence the Fourier transform of $f\tilde{g}$ is well defined and using Fubini's theorem we get

$$\begin{aligned}\widehat{f\tilde{g}}(\xi) = \int f(x)e(-\xi\cdot x)\int g(y)e(y\cdot x)\,\mathrm{d}y\,\mathrm{d}x &= \int g(y)\int f(x)e((y-\xi)\cdot x)\,\mathrm{d}x\,\mathrm{d}y\\ &= \int g(y)\hat{f}(\xi-y)\,\mathrm{d}y\\ &= \hat{f}*g(\xi).\end{aligned}$$

□

Lemma 4.4.2. *Let $f \in L^1(\mathbb{R}^n)$, $Q\subset[0,1]^{n-1}$ be a cube and $g\in L^1(Q)$. Then for all $x\in\mathbb{R}^n$*

$$\widehat{fE_Qg}(x) = \int_Q g(\xi)\widehat{f}(x-\rho(\xi))\,\mathrm{d}\xi, \tag{4.14}$$

where $\rho(\xi) = (\xi_1,\ldots,\xi_{n-1},|\xi|^2)$. If in addition $f\in\mathcal{S}(\mathbb{R}^n)$, then

$$\widetilde{\widehat{fE_Qg}} = fE_Qg. \tag{4.15}$$

Proof. By Fubini's theorem,

$$\begin{aligned}\widehat{fE_Qg}(x) &= \int f(y)e(-x\cdot y)\int_Q g(\xi)e(y\cdot\rho(\xi))\,\mathrm{d}\xi\,\mathrm{d}y\\ &= \int_Q g(\xi)\int f(y)e(-(x-\rho(\xi))\cdot y)\,\mathrm{d}y\,\mathrm{d}\xi\\ &= \int_Q g(\xi)\widehat{f}(x-\rho(\xi))\,\mathrm{d}\xi.\end{aligned}$$

By using this equality, we get by Fubini's theorem and a simple change of variables that

$$
\begin{aligned}
\widehat{\widehat{\widetilde{fE_Qg}}}(y) &= \int e(-y\cdot x)\int_Q g(\xi)\widehat{f}(-x-\rho(\xi))\,\mathrm{d}\xi\,\mathrm{d}x \\
&= \int_Q g(\xi)\int \widehat{f}(-(x+\rho(\xi)))e(-y\cdot x)\,\mathrm{d}x\,\mathrm{d}\xi \\
&= \int_Q g(\xi)\int \widehat{f}(-x)e(-y\cdot x)e(y\cdot\rho(\xi))\,\mathrm{d}x\,\mathrm{d}\xi \\
&= \int_Q g(\xi)e(y\cdot\rho(\xi))\widehat{\widehat{\widetilde{f}}}(y)\,\mathrm{d}\xi \\
&= \int_Q g(\xi)e(y\cdot\rho(\xi))f(y)\,\mathrm{d}\xi \qquad \text{(see 3.3)} \\
&= f(y)E_Qg(y).
\end{aligned}
$$

□

Theorem 4.4.3. *(A reverse Hölder inequality) Let $n\geq 2$. Let $1\leq p\leq q$ and let $E>\frac{nq}{p}$. Let $R\geq 1$. Then for each cube $Q\subset[0,1]^{n-1}$ with $l(Q)=\frac{1}{R}$ and each cube $B\subset\mathbb{R}^n$ with $l(B)=R$ and each $g\in L^1(Q)$ we have*

$$\|E_Qg\|_{L^q_{\#}(w_{B,E})}\lesssim_{E,q,p,n}\|E_Qg\|_{L^p_{\#}(w_{B,\frac{Ep}{q}})}.$$

Especially, the implicit constant is independent of R, Q, B and g.

Proof. Fix E, q, p and R as above. Let $Q\subset[0,1]^{n-1}$ be a cube with $l(Q)=1/R$ and let $g\in L^1(Q)$. Let $B=B(c_B,R)\subset\mathbb{R}^n$ be a cube with $l(B)=R$.

Let $\varphi:\mathbb{R}^n\to[0,\infty)$ be the function constructed in Lemma 3.4.4. Now φ is a Schwartz function satisfying $1_{B(0,1)}\leq\varphi$, such that the support $\widehat{\varphi}$ is contained in the cube $B(0,1)$.

We let $\eta=\varphi^p$. Now since

$$1_B(x)=1_B\Big(\frac{x-c_B}{R}\Big)\leq\varphi\Big(\frac{x-c_B}{R}\Big)=\varphi_B(x)$$

we get

$$\|E_Qg\|_{L^q(B)}\leq\|E_Qg\|_{L^q(\varphi_B^q)}=\|\varphi_BE_Qg\|_{L^q(\mathbb{R}^n)}. \tag{4.16}$$

Also, $\varphi_B:\mathbb{R}^n\to[0,\infty)$ is a Schwartz function and by the properties described in 3.3 we have

$$\widehat{\varphi_B}(x)=e(-c_B\cdot x)R^n\hat{\varphi}(Rx).$$

for all $x\in\mathbb{R}^n$. Thus

$$\operatorname{supp}(\widehat{\varphi_B})\subset B(0,R^{-1}). \tag{4.17}$$

Denote $\rho(\xi) = (\xi_1, \dots, \xi_{n-1}, |\xi|^2)$. Let d be the center of Q. Assume that $\widehat{\varphi_B E_Q g}(x) \neq 0$ for some x. Applying (4.14) we see that then $\widehat{\varphi_B}(x - \rho(\xi)) \neq 0$ for some $\xi \in Q$. This in turn implies by (4.17) that $x \in B(\rho(\xi), R^{-1})$. By applying the triangle inequality we see that $|x_j - d_j| \leq R^{-1}$ for $1 \leq j \leq n-1$. Also,

$$\begin{aligned}
|x_n - |d|^2| &\leq |x_n - |\xi|^2| + ||\xi|^2 - |d|^2| \\
&= |x_n - \rho(\xi)_n| + ||\xi| - |d||(|\xi| + |d|) \\
&\leq \frac{R^{-1}}{2} + |\xi - d|(|\xi| + |d|) \\
&\leq \frac{R^{-1}}{2} + \frac{\sqrt{n-1}R^{-1}}{2} 2\sqrt{n-1} \\
&= \Big(n - \frac{1}{2}\Big) R^{-1} \\
&\leq nR^{-1}.
\end{aligned}$$

Hence $x \in B(\rho(d), 2nR^{-1})$. We denote $S = 2nR^{-1}$. Let $\Delta = \Delta(-\rho(d), S)$ be the cube with center $-\rho(d)$ and side length S. We just showed that

$$\operatorname{supp}(\widehat{\varphi_B E_Q g}) \subset -\Delta. \tag{4.18}$$

Let $\theta : \mathbb{R}^n \to [0, \infty)$ be the Schwartz function constructed in Lemma 3.4.3. Then θ equals to 1 on the cube $B(0, 1)$. Consider the Schwartz function θ_Δ. Observe that if $x \in \Delta$, then $\theta_\Delta(x) = 1$. Thus by (4.18) $\theta_\Delta \equiv 1$ in the support of $\widetilde{\widehat{\varphi_B E_Q g}}$. Using this, Lemma 4.4.1 and (4.15) we get

$$(\varphi_B E_Q g) * \widehat{\theta_\Delta} = \Big(\widetilde{\widehat{\varphi_B E_Q g}}\theta_\Delta\Big)^{\widehat{}} = \widehat{\widetilde{\widehat{\varphi_B E_Q g}}} = \varphi_B E_Q g.$$

Applying Young's convolution inequality (Theorem 3.1.4) we get

$$\begin{aligned}
\|\varphi_B E_Q g\|_{L^q(\mathbb{R}^n)} = \|(\varphi_B E_Q g) * \widehat{\theta_\Delta}\|_{L^q(\mathbb{R}^n)} &\leq \|\varphi_B E_Q g\|_{L^p(\mathbb{R}^n)} \|\widehat{\theta_\Delta}\|_{L^r(\mathbb{R}^n)} \\
&= \|E_Q g\|_{L^p(\eta_B)} \|\widehat{\theta_\Delta}\|_{L^r(\mathbb{R}^n)},
\end{aligned} \tag{4.19}$$

where $r \in [1, \infty)$ is such that

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1 \Leftrightarrow r = \frac{pq}{pq + p - q}.$$

Observe that

$$
\begin{aligned}
\|\widehat{\theta_\Delta}\|_{L^r(\mathbb{R}^n)} &= \Big(\int |S^n\hat\theta(Sx)|^r \,\mathrm{d}x\Big)^{\frac{1}{r}} \\
&= S^n\Big(\int |\hat\theta(Sx)|^r \,\mathrm{d}x\Big)^{\frac{1}{r}} \\
&\lesssim_n R^{-n}\Big(\int |\hat\theta(Sx)|^r \,\mathrm{d}x\Big)^{\frac{1}{r}} \\
&= R^{-n}\Big(\int S^{-n}|\hat\theta(x)|^r \,\mathrm{d}x\Big)^{\frac{1}{r}} \\
&= R^{-n}S^{-\frac{n}{r}}\|\hat\theta\|_{L^r(\mathbb{R}^n)} \\
&\lesssim_{q,p,n} R^{-n}R^{\frac{n}{r}}\|\hat\theta\|_{L^r(\mathbb{R}^n)} \qquad\qquad (\theta \text{ only depends on } n) \\
&\lesssim_{q,p,n} R^{-n(1-\frac{1}{r})}.
\end{aligned}
$$

Combining this with (4.16) and (4.19) we have now reached the following important estimate:

$$
\|E_Q g\|_{L^q(B)} \lesssim_{q,p,n} R^{-n(1-\frac{1}{r})}\|E_Q g\|_{L^p(\eta_B)}. \tag{4.20}
$$

Next we use Lemma 4.2.3 for $F := Ep/q > n$. We define the operators $\mathcal{W} \to [0,\infty]$

$$
O_1(v) := \|E_Q g\|^p_{L^q(v^{q/p})} = \Big(\int ||E_Q g|^p v|^{\frac{q}{p}}\Big)^{\frac{p}{q}} = \||E_Q g|^p v\|_{L^{q/p}(\mathbb{R}^n)}
$$

and

$$
O_2(v) := R^{-np(1-\frac{1}{r})}\|E_Q g\|^p_{L^p(v)}.
$$

Clearly (V4) holds for both O_1 and O_2. Let $\alpha_k \in (0,\infty)$ and $u_k \in \mathcal{W}$ be such that $\sum_{k=1}^\infty \alpha_k u_k \in \mathcal{W}$. Since $q/p \geq 1$, by Minkowski (see Theorem 3.1.3) we get

$$
\begin{aligned}
O_1\Big(\sum_{k=1}^\infty \alpha_k u_k\Big) = \Big\||E_Q g|^p \sum_{k=1}^\infty \alpha_k u_k\Big\|_{L^{q/p}(\mathbb{R}^n)} &\leq \sum_{k=1}^\infty \alpha_k \||E_Q g|^p u_k\|_{L^{q/p}(\mathbb{R}^n)} \\
&= \sum_{k=1}^\infty \alpha_k O_1(u_k).
\end{aligned}
$$

Thus (V2) holds. As seen in the proof of Proposition 4.3.1,

$$
O_2\Big(\sum_{k=1}^\infty \alpha_k u_k\Big) = \sum_{k=1}^\infty \alpha_k O_2(u_k)
$$

since O_2 without the coefficient $R^{-np(1-\frac{1}{r})}$ is of the form stated there and having the coefficient preserves the equality. Thus (V3) holds. Since (4.20) translates to $O_1(1_B)^{1/p} \lesssim_{q,p,n} O_2(\eta_B)^{1/p}$, also (V1) holds. Now Lemma 4.2.3 implies that $O_1(w_{B,F}) \lesssim_{E,p,q,n} O_2(w_{B,F})$ which implies that

$$\|E_Q g\|_{L^q(w_{B,F}^{q/p})} \lesssim_{E,q,p,n} R^{-n(1-\frac{1}{r})} \|E_Q g\|_{L^p(w_{B,F})}.$$

This is equivalent with (note that $1 - \frac{1}{r} = \frac{1}{p} - \frac{1}{q}$)

$$\begin{aligned}
&\|E_Q g\|_{L^q(w_{B,E})} \;\lesssim_{E,q,p,n}\; R^{-n(\frac{1}{p}-\frac{1}{q})}\|E_Q g\|_{L^p(w_{B,F})} \\
\Leftrightarrow\;& \Big(\int |E_Q g|^q w_{B,E}\Big)^{\frac{1}{q}} \;\lesssim_{E,q,p,n}\; R^{-\frac{n}{p}} R^{\frac{n}{q}} \Big(\int |E_Q g|^p w_{B,F}\Big)^{\frac{1}{p}} \\
\Leftrightarrow\;& \Big(\frac{1}{R^n}\int |E_Q g|^q w_{B,E}\Big)^{\frac{1}{q}} \;\lesssim_{E,q,p,n}\; \Big(\frac{1}{R^n}\int |E_Q g|^p w_{B,F}\Big)^{\frac{1}{p}} \\
\Leftrightarrow\;& \|E_Q g\|_{L^q_{\#}(w_{B,E})} \;\lesssim_{E,q,p,n}\; \|E_Q g\|_{L^p_{\#}(w_{B,\frac{Ep}{q}})}.
\end{aligned}$$

$\square$

Chapter 5

Inequalities related to decoupling

In this chapter, we consider propositions that resemble our main decoupling inequality (2.5). The first one will be used in chapter 7 while the other one is useful in the case $p > \frac{2n}{n-1}$ of the l^2 decoupling theorem.

5.1 L^2 decoupling

Proposition 5.1.1. *Fix $E > n$. Let $R \geq 1$. Let $Q \subset [0,1]^{n-1}$ be a cube with $l(Q) \geq R^{-1}$. Then for each cube $B \subset \mathbb{R}^n$ with side length R and each $g \in L^1(Q)$ we have*

$$\|E_Q g\|_{L^2(w_{B,E})} \lesssim_{E,n} \Big(\sum_{q \in \mathrm{Part}_{\frac{1}{R}}(Q)} \|E_q g\|^2_{L^2(w_{B,E})} \Big)^{\frac{1}{2}}. \tag{5.1}$$

Proof. We intend to use Lemma 4.2.3. To this end, let $\varphi : \mathbb{R}^n \to [0,\infty)$ be the function constructed in Lemma 3.4.4. Now φ is a Schwartz function satisfying $1_{B(0,1)} \leq \varphi$, such that the support of the Fourier transform of φ is contained in the cube $B(0,1)$. Let $\eta = \varphi^2$.

Fix $E > n$ and let $Q \subset [0,1]^{n-1}$ be a cube with $l(Q) \geq R^{-1}$, where $R \geq 1$. Let $g \in L^1(Q)$. We define the operators

$$O_1(v) = \|E_Q g\|^2_{L^2(v)}$$

and

$$O_2(v) = \sum_{q \in \mathrm{Part}_{\frac{1}{R}}(Q)} \|E_q g\|^2_{L^2(v)}.$$

By Proposition 4.3.1 these satisfy conditions (V2)-(V4) of Lemma 4.2.3.

Let $B = B(c_B, R) \subset \mathbb{R}^n$ be a cube. By Lemma 4.2.3 the claim follows once we prove that

$$O_1(1_B) \lesssim_n O_2(\eta_B). \tag{5.2}$$

If $x \in B$, then $\frac{x-c_B}{R} \in B(0,1)$. Hence if $x \in B$, then $\eta_B(x) = \eta(\frac{x-c_B}{R}) \geq 1$. Thus we can write

$$\|E_Q g\|_{L^2(B)}^2 = \int |E_Q g|^2 1_B \leq \int |E_Q g|^2 \eta_B = \|\sqrt{\eta_B} E_Q g\|_{L^2(\mathbb{R}^n)}^2 = \|\varphi_B E_Q g\|_{L^2(\mathbb{R}^n)}^2.$$

Observe that $\widehat{\varphi_B}(x) = e(-c_B \cdot x) R^n \hat{\varphi}(Rx)$ for $x \in \mathbb{R}^n$ and thus

$$\operatorname{supp}(\widehat{\varphi_B}) \subset B(0, R^{-1}). \tag{5.3}$$

Throughout, we fix some $x \in \mathbb{R}^n$. Denote $\rho(\xi) = (\xi_1, \ldots, \xi_{n-1}, |\xi|^2)$ for $\xi \in \mathbb{R}^{n-1}$. For $q \in \mathrm{Part}_{\frac{1}{R}}(Q)$, denote $h_q = h_{q,B,g} = \varphi_B E_q g$. We get by Lemma 4.4.2 that

$$\widehat{h_q}(x) = \widehat{\varphi_B E_q g}(x) = \int_q g(\xi) \widehat{\varphi_B}(x - \rho(\xi)) \, \mathrm{d}\xi. \tag{5.4}$$

Let A_n be the maximal number of closed cubes that intersect a given cube, when all the cubes are members of an essential partition of a cube in $\mathbb{R}^n$ into smaller cubes. In other words, given a cube $q \in \mathrm{Part}_{\frac{1}{R}}(Q)$, then A_n is the maximal number of cubes in $\mathrm{Part}_{\frac{1}{R}}(Q)$ that are adjacent to q, including q itself. That is, $A_1 = 3$, $A_2 = 9$, $A_3 = 27$, ..., $A_n = 3^n$.

Let $q = q(c, R^{-1}) \in \mathrm{Part}_{\frac{1}{R}}(Q)$. Assume furthermore that there is an element $\xi \in q$ such that $x - \rho(\xi) \in B(0, R^{-1})$. Let then a be an element of some non-adjacent cube to q, the cube still taken from the partition. Then $a \notin q(c, 3R^{-1})$. That is, there is some $1 \leq j \leq n-1$ such that $|a_j - c_j| > \frac{3}{2} R^{-1}$. Then

$$|x_j - c_j| \leq |x_j - \rho(\xi)_j| + |\rho(\xi)_j - \xi_j| + |\xi_j - c_j| \leq \frac{R^{-1}}{2} + |\xi_j - \xi_j| + \frac{R^{-1}}{2} = R^{-1}$$

and thus

$$|x_j - \rho(a)_j| = |x_j - a_j| \geq |a_j - c_j| - |c_j - x_j| > \frac{3}{2} R^{-1} - R^{-1} = \frac{R^{-1}}{2}.$$

Hence $x - \rho(a) \notin B(0, R^{-1})$.

We just proved that given two non-adjacent cubes $q_1, q_2 \in \mathrm{Part}_{\frac{1}{R}}(Q)$, then $x - \rho(q_k) \subset \mathbb{R}^n \setminus B(0, R^{-1})$ for some $k \in \{1, 2\}$. For if $(x - \rho(q_1)) \cap B(0, R^{-1}) \neq \emptyset$, then the previous deduction with $q = q_1$ shows that $x - \rho(q_2) \subset \mathbb{R}^n \setminus B(0, R^{-1})$. By combining this with (5.3) and (5.4) we see that given two non-adjacent cubes $q_1, q_2 \in \mathrm{Part}_{\frac{1}{R}}(Q)$, then

$$\widehat{h_{q_k}}(x) = 0 \text{ for some } k \in \{1, 2\}. \tag{5.5}$$

We show that

$$\Big| \sum_{q \in \mathrm{Part}_{\frac{1}{R}}(Q)} \widehat{h_q}(x) \Big|^2 \lesssim_n \sum_{q \in \mathrm{Part}_{\frac{1}{R}}(Q)} |\widehat{h_q}(x)|^2. \tag{5.6}$$

Let

$$M = \max_{q \in \mathrm{Part}_{\frac{1}{R}}(Q)} |\widehat{h_q}(x)|.$$

If $M = 0$, then clearly (5.6) holds. On the other hand, if we assume that $M > 0$, then

$$\begin{aligned}
\Big| \sum_{q \in \mathrm{Part}_{\frac{1}{R}}(Q)} \widehat{h_q}(x) \Big|^2 &\leq \Big(\sum_{q \in \mathrm{Part}_{\frac{1}{R}}(Q)} |\widehat{h_q}(x)| \Big)^2 \\
&= \Big(\sum_{k=1}^{A_n} |\widehat{h_{q_k}}(x)| \Big)^2 && \text{by (5.5)} \\
&\leq A_n \sum_{k=1}^{A_n} |\widehat{h_{q_k}}(x)|^2 && \text{(Cauchy-Schwarz)} \\
&\lesssim_n \sum_{k=1}^{A_n} |\widehat{h_{q_k}}(x)|^2 \\
&= \sum_{q \in \mathrm{Part}_{\frac{1}{R}}(Q)} |\widehat{h_q}(x)|^2.
\end{aligned}$$

Finally, since x was arbitrary,

$$\begin{aligned}
\|\sqrt{\eta_B} E_Q g\|^2_{L^2(\mathbb{R}^n)} &= \|\widehat{\sqrt{\eta_B} E_Q g}\|^2_{L^2(\mathbb{R}^n)} && \text{by Theorem 3.1.9 (Plancherel)} \\
&= \| \sum_{q \in \mathrm{Part}_{\frac{1}{R}}(Q)} \widehat{h_q} \|^2_{L^2(\mathbb{R}^n)} \\
&\lesssim_n \sum_{q \in \mathrm{Part}_{\frac{1}{R}}(Q)} \|\widehat{h_q}\|^2_{L^2(\mathbb{R}^n)} && \text{by (5.6)} \\
&= \sum_{q \in \mathrm{Part}_{\frac{1}{R}}(Q)} \|h_q\|^2_{L^2(\mathbb{R}^n)} && \text{by Theorem 3.1.9 (Plancherel)} \\
&= \sum_{q \in \mathrm{Part}_{\frac{1}{R}}(Q)} \|\sqrt{\eta_B} E_q g\|^2_{L^2(\mathbb{R}^n)} \\
&= \sum_{q \in \mathrm{Part}_{\frac{1}{R}}(Q)} \|E_q g\|^2_{L^2(\eta_B)}.
\end{aligned}$$

Thus we have proven (5.2). □

Remark 5.1.2. Note that (5.1) does not directly give information about the decoupling constant $\mathrm{Dec}_n(\delta, 2, E)$ despite its similarities with (2.5). That is because on the right-hand side of (2.5) the side length of B is not the reciprocal of the side length of the partition cube. However, Proposition 5.1.1 will be used in Proposition 7.2.2.

5.2 Passage from E_Q to $E_{[0,1]^{n-1}}$

In Lemma 2.6.1 we defined the decoupling constant $\mathrm{Dec}_n(\delta, p, E)$. In that context, we partitioned $[0,1]^{n-1}$ into cubes of smaller side length. But what if we would instead partition a cube $Q \subsetneq [0,1]^{n-1}$ into smaller cubes and tried to find the constants a that satisfy a similar inequality

$$\|E_Q g\|_{L^p(w_{B,E})} \le a \cdot \Big(\sum_{q \in \mathrm{Part}_{\delta^{\frac{1}{2}}}(Q)} \|E_q g\|^2_{L^p(w_{B,E})} \Big)^{\frac{1}{2}}$$

for some small δ and B with $l(B) = \delta^{-1}$? It turns out that $\mathrm{Dec}_n(\delta, p, E)$ scales nicely in this situation as we shall see in Proposition 5.2.3.

We begin with a lemma.

Remark 5.2.1. Let $n \ge 2$. Let $0 < \sigma < 1$, $\sigma \in 4^{-\mathbb{N}}$. Let $Q \subset [0,1]^{n-1}$ be a cube with $l(Q) = \sigma^{1/2}$. We can write $Q = a + [0, \sigma^{\frac{1}{2}}]^{n-1}$ for some $a \in [0, 1 - \sigma^{\frac{1}{2}}]^{n-1}$. We define the linear operator $T_Q : \mathbb{R}^n \to \mathbb{R}^n$, to be used in this section, by the formula

$$T_Q x := ((x_1 + 2a_1 x_n)\sigma^{\frac{1}{2}}, \dots, (x_{n-1} + 2a_{n-1} x_n)\sigma^{\frac{1}{2}}, x_n \sigma).$$

We will use the shorthand $T = T_Q$.

Lemma 5.2.2. *(a) Let $n \ge 1$. Let $E > n$. Let $\alpha \in 2^{\mathbb{Z}}$. Let $B \subset \mathbb{R}^n$ be a cube with side length R. Then for all $x \in \mathbb{R}^n$*

$$\sum_{\Delta \in \mathcal{F}} w_{\Delta,E}(x) \lesssim_{E,n} w_{B,E}(\alpha^{-1} x),$$

where $\mathcal{F}$ is the unique essential partition of αB using cubes Δ of side length R', where $0 < R' \le \alpha R$.

(b) Let $n \ge 2$. Let $E > 3n$. Let $0 < \delta \le \sigma < 1$, $\delta \in 4^{-\mathbb{N}}$, $\sigma \in 4^{-\mathbb{N}}$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be as in 5.2.1. Let $B \subset \mathbb{R}^n$ be a cube with side length δ^{-1}. Then there exists a finite cover $\mathcal{F}$ of $T(B)$, $\mathcal{F}$ consisting of cubes of side length $\delta^{-1}\sigma$, such that

$$\sum_{\Delta \in \mathcal{F}} w_{\Delta,E}(x) \lesssim_{E,n} w_{B,\frac{E}{3}}(T^{-1} x),$$

for all $x \in \mathbb{R}^n$.

Proof. (a): Notice that $\alpha B = B(\alpha c_B, \alpha R)$. Then by Lemma 4.1.5 we get

$$\sum_{\Delta \in \mathcal{F}} w_{\Delta,E}(x) \lesssim_{E,n} w_{\alpha B,E}(x) = w_{B,E}(\alpha^{-1} x).$$

(b): We use part (a). Note that

$$T^{-1}x := T^{-1}(x) = (\sigma^{-\frac{1}{2}}x_1 - 2a_1\sigma^{-1}x_n, \dots, \sigma^{-\frac{1}{2}}x_{n-1} - 2a_{n-1}\sigma^{-1}x_n, \sigma^{-1}x_n)$$

for $x \in \mathbb{R}^n$.

Observations: When $a \neq 0$, then the element a will cause technical difficulties and we will use the approximation $|a| \leq 1$. If $a = 0$, then the formula for T^{-1} simplifies a bit but the last coordinate in the formula for T^{-1} contains a different power of σ than the rest of the coordinates and even then part (a) does not directly apply. These technical difficulties must be solved.

Throughout the proof we will use the well-known fact that

$$|x| \sim_d |x|_\infty \sim_d \sum_{j=1}^{d} |x_j|. \tag{5.7}$$

for $x \in \mathbb{R}^d$, where $|x|_\infty = \max_{1 \leq j \leq d} |x_j|$.

By triangle inequality and because $a_j \leq 1$ we get

$$T(B) \subset \Big[\prod_{j=1}^{n-1} B((Tc_B)_j, 4\delta^{-1}\sigma^{\frac{1}{2}})\Big] \times B((Tc_B)_n, \delta^{-1}\sigma).$$

We choose $\mathcal{F}$ to be the essential partition of the hyperrectangle

$$\Big[\prod_{j=1}^{n-1} B((Tc_B)_j, 4\delta^{-1}\sigma^{\frac{1}{2}})\Big] \times B((Tc_B)_n, \delta^{-1}\sigma)$$

using cubes of side length $\delta^{-1}\sigma$. Observe that now $(c_\Delta)_n = (Tc_B)_n = \sigma(c_B)_n$ for all $\Delta \in \mathcal{F}$.

We use the notation $y' = (y_1, \dots, y_{n-1})$ for $y \in \mathbb{R}^n$. First we separate the last coordinate:

$$\begin{aligned}
&\sum_{\Delta \in \mathcal{F}} w_{\Delta,E}(x) \\
&= \sum_{\Delta \in \mathcal{F}} \Big(1 + \frac{|x - c_\Delta|}{\delta^{-1}\sigma}\Big)^{-\frac{E}{3}} \Big(1 + \frac{|x - c_\Delta|}{\delta^{-1}\sigma}\Big)^{-\frac{2E}{3}} \\
&\leq \sum_{\Delta \in \mathcal{F}} \Big(1 + \frac{|x' - c'_\Delta|}{\delta^{-1}\sigma}\Big)^{-\frac{E}{3}} \Big(1 + \frac{|x_n - (c_\Delta)_n|}{\delta^{-1}\sigma}\Big)^{-\frac{2E}{3}} \\
&= \Big(1 + \frac{|\sigma^{-1}x_n - (c_B)_n|}{\delta^{-1}}\Big)^{-\frac{2E}{3}} \sum_{\Delta \in \mathcal{F}} \Big(1 + \frac{|x' - c'_\Delta|}{\delta^{-1}\sigma}\Big)^{-\frac{E}{3}}
\end{aligned}$$

Note that $\sigma^{\frac{1}{2}} \in 2^{\mathbb{Z}}$. Thus we may apply part (a) in dimension $n-1$ and continue as follows:

$$
\begin{aligned}
&\overset{(a)}{\lesssim}_{E,n} \Big(1 + \frac{|\sigma^{-1}x_n - (c_B)_n|}{\delta^{-1}}\Big)^{-\frac{2E}{3}} \Big(1 + \frac{|\sigma^{-\frac{1}{2}}x' - \sigma^{-\frac{1}{2}}(Tc_B)'|}{4\delta^{-1}}\Big)^{-\frac{E}{3}} \\
&\lesssim_E \Big(1 + \frac{|\sigma^{-1}x_n - (c_B)_n|}{\delta^{-1}}\Big)^{-\frac{2E}{3}} \Big(1 + \frac{|\sigma^{-\frac{1}{2}}x' - \sigma^{-\frac{1}{2}}(Tc_B)'|}{\delta^{-1}}\Big)^{-\frac{E}{3}} \\
&= \Big(1 + \frac{|\sigma^{-1}x_n - (c_B)_n|}{\delta^{-1}}\Big)^{-\frac{E}{3}} \\
&\quad \cdot \Big(1 + \frac{|\sigma^{-1}x_n - (c_B)_n|}{\delta^{-1}}\Big)^{-\frac{E}{3}} \Big(1 + \frac{|\sigma^{-\frac{1}{2}}x' - \sigma^{-\frac{1}{2}}(Tc_B)'|}{\delta^{-1}}\Big)^{-\frac{E}{3}}.
\end{aligned}
$$

In the last product we have

$$
\begin{aligned}
&\Big(1 + \frac{|\sigma^{-1}x_n - (c_B)_n|}{\delta^{-1}}\Big)^{-\frac{E}{3}} \Big(1 + \frac{|\sigma^{-\frac{1}{2}}x' - \sigma^{-\frac{1}{2}}(Tc_B)'|}{\delta^{-1}}\Big)^{-\frac{E}{3}} \\
&\lesssim_E \Big(1 + \frac{2|\sigma^{-1}x_n - (c_B)_n||a|}{\delta^{-1}}\Big)^{-\frac{E}{3}} \Big(1 + \frac{|\sigma^{-\frac{1}{2}}x' - \sigma^{-\frac{1}{2}}(Tc_B)'|}{\delta^{-1}}\Big)^{-\frac{E}{3}} \\
&= \Big(1 + \frac{|2(\sigma^{-1}x_n - (c_B)_n)a|}{\delta^{-1}}\Big)^{-\frac{E}{3}} \Big(1 + \frac{|\sigma^{-\frac{1}{2}}x' - c_B' - 2(c_B)_n a|}{\delta^{-1}}\Big)^{-\frac{E}{3}} \\
&\le \Big(1 + \frac{\max\{|2(\sigma^{-1}x_n - (c_B)_n)a|, |\sigma^{-\frac{1}{2}}x' - c_B' - 2(c_B)_n a|\}}{\delta^{-1}}\Big)^{-\frac{E}{3}} \\
&\overset{(5.7)}{\sim}_E \Big(1 + \frac{|\sigma^{-\frac{1}{2}}x' - c_B' - 2(c_B)_n a| + |2(\sigma^{-1}x_n - (c_B)_n)a|}{\delta^{-1}}\Big)^{-\frac{E}{3}} \\
&\le \Big(1 + \frac{|\sigma^{-\frac{1}{2}}x' - c_B' - 2(c_B)_n a - 2(\sigma^{-1}x_n - (c_B)_n)a|}{\delta^{-1}}\Big)^{-\frac{E}{3}} \\
&= \Big(1 + \frac{|\sigma^{-\frac{1}{2}}x' - 2\sigma^{-1}x_n a - c_B'|}{\delta^{-1}}\Big)^{-\frac{E}{3}} \\
&= \Big(1 + \frac{|(T^{-1}x - c_B)'|}{\delta^{-1}}\Big)^{-\frac{E}{3}}.
\end{aligned}
$$

Put together, we get

$$
\begin{aligned}
&\sum_{\Delta\in\mathcal{F}} w_{\Delta,E}(x)\\
&\lesssim_{E,n} \Big(1+\frac{|\sigma^{-1}x_n-(c_B)_n|}{\delta^{-1}}\Big)^{-\frac{E}{3}}\Big(1+\frac{|(T^{-1}x-c_B)'|}{\delta^{-1}}\Big)^{-\frac{E}{3}}\\
&=\Big(1+\frac{|(T^{-1}x-c_B)_n|}{\delta^{-1}}\Big)^{-\frac{E}{3}}\Big(1+\frac{|(T^{-1}x-c_B)'|}{\delta^{-1}}\Big)^{-\frac{E}{3}}\\
&\le \Big(1+\frac{|T^{-1}x-c_B|_\infty}{\delta^{-1}}\Big)^{-\frac{E}{3}}\\
&\overset{(5.7)}{\sim}_{E,n} \Big(1+\frac{|T^{-1}x-c_B|}{\delta^{-1}}\Big)^{-\frac{E}{3}}\\
&= w_{B,\frac{E}{3}}(T^{-1}x).
\end{aligned}
$$

□

Proposition 5.2.3. *Let $E>n$ and let $p\ge 2$. Let $0<\delta\le\sigma<1$, $\delta\in 4^{-\mathbb{N}}$, $\sigma\in 4^{-\mathbb{N}}$. For each cube $Q\subset[0,1]^{n-1}$ with $l(Q)=\sigma^{1/2}$, each $g\in L^1(Q)$ and each cube $B\subset\mathbb{R}^n$ with $l(B)\ge\delta^{-1}$ we have*

$$\|E_Q g\|_{L^p(w_{B,E})}\lesssim_{E,p,n}\mathrm{Dec}_n\Big(\frac{\delta}{\sigma},p,3E\Big)\Big(\sum_{q\in\mathrm{Part}_{\delta^{\frac12}}(Q)}\|E_q g\|^2_{L^p(w_{B,E})}\Big)^{\frac12}.$$

Proof. Fix $E>n$ and $p\ge 2$. Set $F:=3E$.

Let $0<\delta\le\sigma<1$, $\delta\in 4^{-\mathbb{N}}$, $\sigma\in 4^{-\mathbb{N}}$. Let $Q\subset[0,1]^{n-1}$ be a cube with $l(Q)=\sigma^{1/2}$. We can write $Q=a+[0,\sigma^{\frac12}]^{n-1}$ for some $a=(a_1,\dots,a_{n-1})$. Fix $g\in L^1(Q)$.

(Observe that now $\frac{\delta}{\sigma}\in 4^{-\mathbb{N}}$.)

We define the following operators in the spirit of Proposition 4.3.1:

$$
\begin{aligned}
O_1(v)&=\|E_Q g\|^p_{L^p(v)},\\
O_2(v)&=\mathrm{Dec}\Big(\frac{\delta}{\sigma},p,F\Big)^p\Big(\sum_{q\in\mathrm{Part}_{\delta^{\frac12}}(Q)}\|E_q g\|^2_{L^p(v)}\Big)^{\frac{p}{2}}.
\end{aligned}
$$

By Proposition 4.3.1 these operators satisfy the conditions (W2)-(W4) of Lemma 4.2.2. The operator O_2 is *exactly* of the form discussed in 4.3.1 if the term $\mathrm{Dec}\big(\frac{\delta}{\sigma},p,F\big)^p$ is removed. But that term does not depend on the function v and thus one can check that having it as a multiplier does not affect the satisfaction of the conditions (W3)-(W4). Also, O_1 and O_2 are finite-valued.

We first assume that B is a cube such that $l(B) = \delta^{-1}$. In the light of Lemma 4.2.2 we will show that then

$$\|E_Q g\|_{L^p(B)} \lesssim_{E,p,n} \mathrm{Dec}\Big(\frac{\delta}{\sigma}, p, F\Big)\Big(\sum_{q \in \mathrm{Part}_{\delta^{\frac{1}{2}}}(Q)} \|E_q g\|^2_{L^p(w_{B,E})}\Big)^{\frac{1}{2}}. \tag{5.8}$$

Using our notation, the above is equivalent with

$$O_1(1_B) \lesssim_{E,p,n} O_2(w_B).$$

We define the transformation $L = L_Q$,

$$L(\xi_1, \ldots, \xi_{n-1}) = \Big(\frac{\xi_1 - a_1}{\sigma^{\frac{1}{2}}}, \ldots, \frac{\xi_{n-1} - a_{n-1}}{\sigma^{\frac{1}{2}}}\Big).$$

We point out that $L : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ has an inverse: $L^{-1}(\xi_1, \ldots, \xi_{n-1}) = (\sigma^{\frac{1}{2}}\xi_1 + a_1, \ldots, \sigma^{\frac{1}{2}}\xi_{n-1} + a_{n-1})$. Observe that L is a combination of a translation by $-a$ and a stretching by $\sigma^{-\frac{1}{2}}$. Notice that $L(Q) = [0,1]^{n-1}$.

Let $T = T_Q$ be the transformation defined as

$$T(x_1, \ldots, x_{n-1}, x_n) = ((x_1 + 2a_1x_n)\sigma^{\frac{1}{2}}, \ldots, (x_{n-1} + 2a_{n-1}x_n)\sigma^{\frac{1}{2}}, x_n\sigma).$$

We point out that $T : \mathbb{R}^n \to \mathbb{R}^n$ has an inverse: $T^{-1}(x_1, \ldots, x_{n-1}, x_n) = (x_1\sigma^{-\frac{1}{2}} - 2a_1x_n\sigma^{-1}, \ldots, x_{n-1}\sigma^{-\frac{1}{2}} - 2a_{n-1}x_n\sigma^{-1}, x_n\sigma^{-1})$.

Let $\widetilde{Q} \subset Q$ be a cube. We denote $\widetilde{Q}_L = L(\widetilde{Q})$ and $g_L = g \circ L^{-1} : [0,1]^{n-1} \to \mathbb{C}$. Observe that $\widetilde{Q}_L$ is a cube. Additionally, we denote $\rho(\xi) := (\xi_1, \ldots, \xi_{n-1}, \xi_1^2 + \ldots + \xi_{n-1}^2)$. Now for each $x \in \mathbb{R}^n$

$$\begin{aligned}&|E_{\widetilde{Q}} g(x)| \\ &= \Big|\int_{\widetilde{Q}} g(\xi) e(\rho(\xi) \cdot x)\,\mathrm{d}\xi\Big|.\end{aligned}$$

We show that $|E_{\widetilde{Q}} g(x)| = \sigma^{\frac{n-1}{2}} |E_{\widetilde{Q}_L} g_L(T(x))|$. By change of variables (Theorem 3.1.7),

$$\int_{\widetilde{Q}} g(\xi) e(\rho(\xi) \cdot x)\,\mathrm{d}\xi = \sigma^{\frac{n-1}{2}} \int_{\widetilde{Q}_L} g(L^{-1}(\xi)) e(\rho(L^{-1}(\xi)) \cdot x)\,\mathrm{d}\xi.$$

The justification for the change of variables above is: L^{-1} is a bijection $\widetilde{Q}_L \to \widetilde{Q}$. Both $\widetilde{Q}_L$ and $\widetilde{Q}$ are closed cubes, whose boundaries have zero measure, and the boundaries map to each other. Thus on both sides we can integrate over the interior of a closed cube. The partial derivatives of the component functions of L^{-1} are continuous; thus L^{-1} is differentiable. Observe that $\det(L^{-1})'(x) = (\sigma^{\frac{1}{2}})^{n-1} = \sigma^{\frac{n-1}{2}}$, since $(L^{-1})'(x)$ is an upper triangular $(n-1)\times(n-1)$-matrix. Also, the integral $\int_{\widetilde{Q}} g(\xi)e(\rho(\xi)\cdot x)\,\mathrm{d}\xi$ exists.

We write the term in the right-side integral in a different way:

$$\begin{aligned}
&e(\rho(L^{-1}(\xi))\cdot x)\\
&= e\Big(\Big(\sigma^{\frac{1}{2}}\xi_1 + a_1, \dots, \sigma^{\frac{1}{2}}\xi_{n-1} + a_{n-1}, \sum_{j=1}^{n-1}(\sigma^{\frac{1}{2}}\xi_j + a_i)^2\Big)\cdot x\Big)\\
&= e\Big(\Big(\sigma^{\frac{1}{2}}\xi_1 + a_1, \dots, \sigma^{\frac{1}{2}}\xi_{n-1} + a_{n-1}, \sum_{j=1}^{n-1}\sigma\xi_j^2 + \sum_{j=1}^{n-1}2\sigma^{\frac{1}{2}}\xi_j a_j + \sum_{j=1}^{n-1}a_j^2\Big)\cdot x\Big)\\
&= e\Big(\Big(\sigma^{\frac{1}{2}}\xi_1, \dots, \sigma^{\frac{1}{2}}\xi_{n-1}, \sum_{j=1}^{n-1}\sigma\xi_j^2 + \sum_{j=1}^{n-1}2\sigma^{\frac{1}{2}}\xi_j a_j\Big)\cdot x\Big)e\Big(\rho(a)\cdot x\Big)\\
&= e\Big(\sum_{j=1}^{n-1}\sigma^{\frac{1}{2}}\xi_j x_j + x_n\sum_{j=1}^{n-1}2\sigma^{\frac{1}{2}}\xi_j a_j + x_n\sum_{j=1}^{n-1}\sigma\xi_j^2\Big)e\Big(\rho(a)\cdot x\Big)\\
&= e\Big(\sum_{j=1}^{n-1}\xi_j(x_j + 2a_j x_n)\sigma^{\frac{1}{2}} + x_n\sigma\sum_{j=1}^{n-1}\xi_j^2\Big)e\Big(\rho(a)\cdot x\Big)\\
&= e(\rho(\xi)\cdot T(x))e(\rho(a)\cdot x).
\end{aligned}$$

We have shown that $g(L^{-1}(\xi))e(\rho(L^{-1}(\xi))\cdot x) = g(L^{-1}(\xi))e(\rho(\xi)\cdot T(x))e(\rho(a)\cdot x)$.

Thus

$$\begin{aligned}
\Big|\int_{\widetilde{Q}} g(\xi)e(\rho(\xi)\cdot x)\,\mathrm{d}\xi\Big| &= \Big|\sigma^{\frac{n-1}{2}}\int_{\widetilde{Q}_L} g(L^{-1}(\xi))e(\rho(\xi)\cdot T(x))e(\rho(a)\cdot x)\,\mathrm{d}\xi\Big|\\
&= \sigma^{\frac{n-1}{2}}\Big|e(\rho(a)\cdot x)\Big|\Big|\int_{\widetilde{Q}_L} g(L^{-1}(\xi))e(\rho(\xi)\cdot T(x))\,\mathrm{d}\xi\Big|\\
&= \sigma^{\frac{n-1}{2}}\Big|\int_{\widetilde{Q}_L} g(L^{-1}(\xi))e(\rho(\xi)\cdot T(x))\,\mathrm{d}\xi\Big|\\
&= \sigma^{\frac{n-1}{2}}|E_{\widetilde{Q}_L}g_L(T(x))|
\end{aligned}$$

i.e.

$$|E_{\widetilde{Q}}g(x)| = \sigma^{\frac{n-1}{2}}|E_{\widetilde{Q}_L}g_L(T(x))| \tag{5.9}$$

for all $x\in\mathbb{R}^n$, which is what we wanted to show.

We denote $S = T(B)$. We let $\mathcal{F}$ be the cover of S described in Lemma 5.2.2. Now $\mathcal{F}$ consists of cubes of side length $\delta^{-1}\sigma$. A similar proof as in 4.1.1 yields

$$1_S(x) \lesssim_{E,n} \sum_{\Delta\in\mathcal{F}} w_{\Delta,F}(x).$$

By Lemma 5.2.2 we have

$$\sum_{\Delta\in\mathcal{F}} w_{\Delta,F}(x) \lesssim_{E,n} w_{B,E}(T^{-1}x).$$

Joining these we have

$$1_S(x) \lesssim_{E,n} \sum_{\Delta \in \mathcal{F}} w_{\Delta,F}(x) \lesssim_{E,n} w_{B,E}(T^{-1}x). \tag{5.10}$$

Hence (remember that $Q_L = L(Q) = [0,1]^{n-1}$)

$$\begin{aligned}
\|E_Q g\|_{L^p(B)} &= \Big(\int_B |E_Q g(x)|^p \,\mathrm{d}x \Big)^{\frac{1}{p}} \\
&= \Big(\int_B \sigma^{\frac{p(n-1)}{2}} |E_{Q_L} g_L(T(x))|^p \,\mathrm{d}x \Big)^{\frac{1}{p}} && \text{by (5.9)} \\
&= \sigma^{\frac{n-1}{2}} \Big(\int_B |E g_L(T(x))|^p \,\mathrm{d}x \Big)^{\frac{1}{p}} \\
&\text{(change of variables)} \\
&= \sigma^{\frac{n-1}{2}} \Big(\sigma^{-\frac{n+1}{2}} \int_S |E g_L(x)|^p \,\mathrm{d}x \Big)^{\frac{1}{p}} \\
&= \sigma^{\frac{n-1}{2}} \sigma^{-\frac{n+1}{2p}} \|E g_L\|_{L^p(S)}.
\end{aligned}$$

The justification for the change of variables above is: T is a bijection $B \to S$. By Remark 3.1.8 we can consider the integrals over the interior of B and the image of that interior. The partial derivatives of the component functions of T are continuous; thus T is differentiable. Observe that $\det T'(x) = (\sigma^{\frac{1}{2}})^{n-1}\sigma = \sigma^{\frac{n+1}{2}}$, since $T'(x)$ is an upper triangular $n \times n$-matrix.

By (5.10) we get

$$\begin{aligned}
\|E g_L\|_{L^p(S)} &= \Big(\int_{\mathbb{R}^n} |E g_L(x)|^p 1_S(x) \,\mathrm{d}x \Big)^{\frac{1}{p}} \\
&\lesssim_{E,p,n} \Big(\sum_{\Delta \in \mathcal{F}} \int_{\mathbb{R}^n} |E g_L(x)|^p w_{\Delta,F}(x) \,\mathrm{d}x \Big)^{\frac{1}{p}} \\
&= \Big(\sum_{\Delta \in \mathcal{F}} \|E g_L\|^p_{L^p(w_{\Delta,F})} \Big)^{\frac{1}{p}},
\end{aligned}$$

and by the definition of the decoupling constant, this is dominated as follows (it is essential that $l(\Delta) = (\delta/\sigma)^{-1}$) :

$$\begin{aligned}
&\leq \Big[\sum_{\Delta \in \mathcal{F}} \mathrm{Dec}\Big(\frac{\delta}{\sigma}, p, F\Big)^p \Big(\sum_{q' \in \mathrm{Part}_{(\frac{\delta}{\sigma})^{\frac{1}{2}}}([0,1]^{n-1})} \|E_{q'} g_L\|^2_{L^p(w_{\Delta,F})} \Big)^{\frac{p}{2}} \Big]^{\frac{1}{p}} \\
&= \mathrm{Dec}\Big(\frac{\delta}{\sigma}, p, F\Big) \Big[\sum_{\Delta \in \mathcal{F}} \Big(\sum_{q' \in \mathrm{Part}_{(\frac{\delta}{\sigma})^{\frac{1}{2}}}([0,1]^{n-1})} \|E_{q'} g_L\|^2_{L^p(w_{\Delta,F})} \Big)^{\frac{p}{2}} \Big]^{\frac{1}{p}}.
\end{aligned}$$

Observe that $q \in \mathrm{Part}_{\delta^{\frac{1}{2}}}(Q)$ if and only if $q = \prod_{k=1}^{n-1}(a_k + [(j_k - 1)\delta^{\frac{1}{2}}, j_k\delta^{\frac{1}{2}}])$ for some $j_k \in \{1, 2, \ldots, (\frac{\sigma}{\delta})^{\frac{1}{2}}\}$, $1 \le k \le n-1$. Thus if $q \in \mathrm{Part}_{\delta^{\frac{1}{2}}}(Q)$, then

$$q_L = L(q) = \prod_{k=1}^{n-1}[(j_k - 1)\Big(\frac{\delta}{\sigma}\Big)^{\frac{1}{2}}, j_k\Big(\frac{\delta}{\sigma}\Big)^{\frac{1}{2}}].$$

Since

$$\mathrm{Part}_{(\frac{\delta}{\sigma})^{\frac{1}{2}}}([0,1]^{n-1}) = \Big\{\prod_{k=1}^{n-1}[(j_k - 1)\Big(\frac{\delta}{\sigma}\Big)^{\frac{1}{2}}, j_k\Big(\frac{\delta}{\sigma}\Big)^{\frac{1}{2}}] \colon \forall k \ j_k \in \{1, 2, \ldots, (\frac{\sigma}{\delta})^{\frac{1}{2}}\}\Big\},$$

we have shown that

$$\{q_L : q \in \mathrm{Part}_{\delta^{\frac{1}{2}}}(Q)\} = \mathrm{Part}_{(\frac{\delta}{\sigma})^{\frac{1}{2}}}([0,1]^{n-1})$$

In addition, $(q_1)_L \neq (q_2)_L$ if $q_1, q_2 \in \mathrm{Part}_{\delta^{\frac{1}{2}}}(Q)$ and $q_1 \neq q_2$. Hence for each $\Delta \in \mathcal{F}$

$$\sum_{q' \in \mathrm{Part}_{(\frac{\delta}{\sigma})^{\frac{1}{2}}}([0,1]^{n-1})} \|E_{q'} g_L\|^2_{L^p(w_{\Delta,F})} = \sum_{q \in \mathrm{Part}_{\delta^{\frac{1}{2}}}(Q)} \|E_{q_L} g_L\|^2_{L^p(w_{\Delta,F})}.$$

By the reverse Minkowski inequality in $l_{\frac{2}{p}}$

$$\begin{aligned}
\sum_{\Delta \in \mathcal{F}} \Big(\sum_{q \in \mathrm{Part}_{\delta^{\frac{1}{2}}}(Q)} \|E_{q_L} g_L\|^2_{L^p(w_{\Delta,F})}\Big)^{\frac{p}{2}} &= \sum_{\Delta \in \mathcal{F}} \Big(\sum_{q \in \mathrm{Part}_{\delta^{\frac{1}{2}}}(Q)} (\|E_{q_L} g_L\|^p_{L^p(w_{\Delta,F})})^{\frac{2}{p}}\Big)^{\frac{p}{2}} \\
&\le \Big(\sum_{q \in \mathrm{Part}_{\delta^{\frac{1}{2}}}(Q)} \Big(\sum_{\Delta \in \mathcal{F}} \|E_{q_L} g_L\|^p_{L^p(w_{\Delta,F})}\Big)^{\frac{2}{p}}\Big)^{\frac{p}{2}} \\
&= \Big(\sum_{q \in \mathrm{Part}_{\delta^{\frac{1}{2}}}(Q)} \Big(\|E_{q_L} g_L\|^p_{L^p(\sum w_{\Delta,F})}\Big)^{\frac{2}{p}}\Big)^{\frac{p}{2}} \\
&= \Big(\sum_{q \in \mathrm{Part}_{\delta^{\frac{1}{2}}}(Q)} \|E_{q_L} g_L\|^2_{L^p(\sum w_{\Delta,F})}\Big)^{\frac{p}{2}}.
\end{aligned}$$

Then by combining the above and followed by (5.10) we finally get (we denote

$w_B = w_{B,E}$)

$$\begin{aligned}\|E_Q g\|_{L^p(B)} &\lesssim_{E,p,n} \sigma^{\frac{n-1}{2}}\sigma^{-\frac{n+1}{2p}}\operatorname{Dec}\Big(\frac{\delta}{\sigma},p,F\Big)\Big[\Big(\sum_{q\in\operatorname{Part}_{\delta^{\frac12}}(Q)}\|E_{q_L}g_L\|^2_{L^p(\sum w_{\Delta,F})}\Big)^{\frac{p}{2}}\Big]^{\frac1p}\\ &= \sigma^{\frac{n-1}{2}}\sigma^{-\frac{n+1}{2p}}\operatorname{Dec}\Big(\frac{\delta}{\sigma},p,F\Big)\Big(\sum_{q\in\operatorname{Part}_{\delta^{\frac12}}(Q)}\|E_{q_L}g_L\|^2_{L^p(\sum w_{\Delta,F})}\Big)^{\frac12}\\ &\lesssim_{E,p,n} \sigma^{\frac{n-1}{2}}\sigma^{-\frac{n+1}{2p}}\operatorname{Dec}\Big(\frac{\delta}{\sigma},p,F\Big)\Big(\sum_{q\in\operatorname{Part}_{\delta^{\frac12}}(Q)}\|E_{q_L}g_L\|^2_{L^p(w_B\circ T^{-1})}\Big)^{\frac12}\\ &= \operatorname{Dec}\Big(\frac{\delta}{\sigma},p,F\Big)\Big(\sum_{q\in\operatorname{Part}_{\delta^{\frac12}}(Q)}\Big(\sigma^{\frac{n-1}{2}}\sigma^{-\frac{n+1}{2p}}\|E_{q_L}g_L\|_{L^p(w_B\circ T^{-1})}\Big)^2\Big)^{\frac12}\end{aligned}$$

Then by change of variables, when $q \in \operatorname{Part}_{\delta^{\frac12}}(Q)$, then

$$\begin{aligned}\sigma^{\frac{n-1}{2}}\sigma^{-\frac{n+1}{2p}}\|E_{q_L}g_L\|_{L^p(w_B\circ T^{-1})} &= \sigma^{\frac{n-1}{2}}\sigma^{-\frac{n+1}{2p}}\Big(\int_{\mathbb{R}^n}|E_{q_L}g_L(x)|^p w_B(T^{-1}(x))\,\mathrm{d}x\Big)^{\frac1p}\\ &= \sigma^{\frac{n-1}{2}}\Big(\int_{\mathbb{R}^n}\sigma^{-\frac{n+1}{2}}|E_{q_L}g_L(x)|^p w_B(T^{-1}(x))\,\mathrm{d}x\Big)^{\frac1p}\\ &\text{(change of variables)}\\ &= \sigma^{\frac{n-1}{2}}\Big(\int_{\mathbb{R}^n}|E_{q_L}g_L(T(x))|^p w_B(x)\,\mathrm{d}x\Big)^{\frac1p}\\ &= \Big(\int_{\mathbb{R}^n}|E_q g(x)|^p w_B(x)\,\mathrm{d}x\Big)^{\frac1p} \qquad \text{by (5.9)}\\ &= \|E_q g\|_{L^p(w_B)}\end{aligned}$$

The change of variables is justified because T is a bijection $\mathbb{R}^n \to \mathbb{R}^n$ and because of the reasons already stated in the previous change of variables.

Finally, we have

$$\|E_Q g\|_{L^p(B)} \lesssim_{E,p,n} \operatorname{Dec}\Big(\frac{\delta}{\sigma},p,F\Big)\Big(\sum_{q\in\operatorname{Part}_{\delta^{\frac12}}(Q)}\|E_q g\|^2_{L^p(w_{B,E})}\Big)^{\frac12}, \tag{5.11}$$

which is the estimate (5.8) that we wanted to show. Now by Lemma 4.2.2

$$O_1(w_{B,E}) \lesssim_{E,p,n} O_2(w_{B,E}),$$

because the implicit constant in (5.11) only depends on E, p and n. This finishes the proof in the case $l(B) = \delta^{-1}$ after taking the pth root of both sides.

Fix next some $\tau \geq \delta^{-1}$ and assume that $l(B) = \tau$. (Remember that $\delta^{-1} > 1$.) Here all weight functions are with respect to E and so we omit E from their subscripts. Then

$$\begin{aligned}
\|E_Q g\|_{L^p(B)} &= \Big(\int_{\mathbb{R}^n} |E_Q g|^p 1_B\Big)^{\frac{1}{p}} \\
&\lesssim_{E,p,n} \Big(\int_{\mathbb{R}^n} |E_Q g|^p \sum_{\Delta\in \mathrm{Part}_{\delta^{-1}}(B)} w_\Delta\Big)^{\frac{1}{p}} && \text{(Lemma 4.1.5)} \\
&= \Big(\sum_{\Delta\in \mathrm{Part}_{\delta^{-1}}(B)} \int_{\mathbb{R}^n} |E_Q g|^p w_\Delta\Big)^{\frac{1}{p}} \\
&= \Big(\sum_{\Delta\in \mathrm{Part}_{\delta^{-1}}(B)} \|E_Q g\|^p_{L^p(w_\Delta)}\Big)^{\frac{1}{p}} \\
&\lesssim_{E,p,n} \Big(\sum_{\Delta\in \mathrm{Part}_{\delta^{-1}}(B)} \mathrm{Dec}\Big(\frac{\delta}{\sigma}, p, F\Big)^p \Big(\sum_{q\in \mathrm{Part}_{\delta^{\frac{1}{2}}}(Q)} \|E_q g\|^2_{L^p(w_\Delta)}\Big)^{\frac{p}{2}}\Big)^{\frac{1}{p}} \\
&= \mathrm{Dec}\Big(\frac{\delta}{\sigma}, p, F\Big)\Big(\sum_{\Delta\in \mathrm{Part}_{\delta^{-1}}(B)} \Big(\sum_{q\in \mathrm{Part}_{\delta^{\frac{1}{2}}}(Q)} \|E_q g\|^2_{L^p(w_\Delta)}\Big)^{\frac{p}{2}}\Big)^{\frac{1}{p}},
\end{aligned}$$

where the second $\lesssim$ holds since $l(\Delta) = \delta^{-1}$ (we use the first part of this proof). It is crucial there that the implicit constant is independent of Δ and therefore can be taken out of the sum $\sum_{\Delta\in \mathrm{Part}_{\delta^{-1}}(B)}$.

By using the reverse Minkowski inequality just like in the first part of the proof we deduce that the above is dominated by

$$\begin{aligned}
&\le \mathrm{Dec}\Big(\frac{\delta}{\sigma}, p, F\Big)\Big(\sum_{q\in \mathrm{Part}_{\delta^{\frac{1}{2}}}(Q)} \|E_q g\|^2_{L^p(\sum w_\Delta)}\Big)^{\frac{1}{2}} \\
&\lesssim_{E,p,n} \mathrm{Dec}\Big(\frac{\delta}{\sigma}, p, F\Big)\Big(\sum_{q\in \mathrm{Part}_{\delta^{\frac{1}{2}}}(Q)} \|E_q g\|^2_{L^p(w_B)}\Big)^{\frac{1}{2}}. && \text{(Lemma 4.1.5)}
\end{aligned}$$

We have proved that for all cubes $B \subset \mathbb{R}^n$ of side length τ it holds that $O_1(1_B) \lesssim_{E,p,n} O_2(w_B)$. Thus by Lemma 4.2.2 $O_1(w_B) \lesssim_{E,p,n} O_2(w_B)$ for all cubes $B \subset \mathbb{R}^n$ with side length τ, which finishes the proof for side length τ after taking the pth root of both sides. (Again the constant does not depend on τ.)

We split the proof into two cases; the final implicit constant is taken to be the maximum of the two implicit constants. The claim is proved. □

Remark 5.2.4. Proposition 5.2.3 will not be used in this thesis. It will, however, be helpful in the case $p > \frac{2n}{n-1}$ of the l^2 decoupling theorem. For this, see [3].

Chapter 6

The multilinear decoupling constant

In this chapter, we will present the so-called multilinear decoupling constant $\mathrm{Dec}_n(\delta, p, \nu, m, E)$. This depends on the concept of ν-transversality, which is defined shortly. In Theorem 6.2.1 we will present a result that shows that the decoupling constants $\mathrm{Dec}_n(\delta, p, \nu, m, E)$ dominate the decoupling constants $\mathrm{Dec}_n(\delta, p, E)$.

6.1 The decoupling constant with respect to m and ν

Denote $\mathbb{P}^{n-1} := \{(\xi_1, \ldots, \xi_{n-1}, \xi_1^2 + \ldots + \xi_{n-1}^2) \colon 0 \le \xi_i \le 1\}$. Let $\pi : \mathbb{P}^{n-1} \to [0,1]^{n-1}$ be the projection map, that is, $\pi(\xi_1, \ldots, \xi_{n-1}, \xi_1^2 + \ldots + \xi_{n-1}^2) = (\xi_1, \ldots, \xi_{n-1})$.

Definition 6.1.1. Let $0 < \nu < 1$. We say that the cubes $Q_1, \ldots, Q_n \subset [0,1]^{n-1}$ are ν-transverse if the volume of the parallelepiped spanned by unit normals $n(P_i)$ is greater or equal to ν, for each choice of $P_i \in \mathbb{P}^{n-1}$ with $\pi(P_i) \in Q_i$.

Definition 6.1.2. Let $n \ge 2$, $E > n$, $p \ge 2$, $m \in \mathbb{N}_+$, $0 < \nu < 1$ and $\delta \in 4^{-\mathbb{N}}$.

We let $\mathrm{Dec}_n(\delta, p, \nu, m, E)$ be the smallest constant $0 \le a < \infty$, such that the inequality

$$\Big[\sum_{\Delta \in \mathrm{Part}_{\mu^{-1}}(B)} \Big(\prod_{i=1}^{n} \|E_{Q_i} g\|_{L^p(w_{\Delta,10E})}^p \Big)^{\frac{1}{n}} \Big]^{\frac{1}{p}}$$
$$\le a \cdot \Big[\prod_{i=1}^{n} \sum_{q_i \in \mathrm{Part}_{\delta^{\frac{1}{2}}}(Q_i)} \|E_{q_i} g\|_{L^p(w_{B,E})}^2 \Big]^{\frac{1}{2n}}$$

holds for each cube $B \subset \mathbb{R}^n$ with $l(B) = \delta^{-1}$, each $g \in L^1([0,1]^{n-1})$ and all ν-transverse cubes $Q_i \subset [0,1]^{n-1}$ with equal side lengths μ satisfying $\mu \ge \delta^{2^{-m}}$.

Remark 6.1.3. Some remarks considering the definition of $\mathrm{Dec}_n(\delta, p, \nu, m, E)$: The existence of the constant is proved shortly. The lower bound $\delta^{2^{-m}}$ imposed on μ is more severe than the lower bound $\delta^{\frac{1}{2}}$ that is needed in order to $\mathrm{Part}_{\delta^{\frac{1}{2}}}(Q_i)$ make sense. The exponent $10E$ on the left-hand side is chosen rather than E for technical reasons and it will affect the proof of Lemma 8.1.1.

Proof of existence: We prove that there exists a constant a that satisfies the condition in Definition 6.1.2. We apply n-product Hölder's inequality (Theorem 3.1.2), Lemma 4.1.5 and (2.4). We get

$$\begin{aligned}
&\Big[\sum_{\Delta\in\mathrm{Part}_{\mu^{-1}}(B)}\Big(\prod_{i=1}^{n}\|E_{Q_i}g\|^p_{L^p(w_{\Delta,10E})}\Big)^{\frac{1}{n}}\Big]^{\frac{1}{p}}\\
&\le\Big[\prod_{i=1}^{n}\Big(\sum_{\Delta\in\mathrm{Part}_{\mu^{-1}}(B)}\|E_{Q_i}g\|^p_{L^p(w_{\Delta,10E})}\Big)^{\frac{1}{n}}\Big]^{\frac{1}{p}}\\
&=\Big[\prod_{i=1}^{n}\Big(\sum_{\Delta\in\mathrm{Part}_{\mu^{-1}}(B)}\|E_{Q_i}g\|^p_{L^p(w_{\Delta,10E})}\Big)^{\frac{1}{p}}\Big]^{\frac{1}{n}}\\
&=\Big[\prod_{i=1}^{n}\Big(\sum_{\Delta\in\mathrm{Part}_{\mu^{-1}}(B)}\int|E_{Q_i}g|^p w_{\Delta,10E}\Big)^{\frac{1}{p}}\Big]^{\frac{1}{n}}\\
&=\Big[\prod_{i=1}^{n}\Big(\int|E_{Q_i}g|^p\sum_{\Delta\in\mathrm{Part}_{\mu^{-1}}(B)}w_{\Delta,10E}\Big)^{\frac{1}{p}}\Big]^{\frac{1}{n}}\\
&\lesssim_{E,p,n}\Big[\prod_{i=1}^{n}\Big(\int|E_{Q_i}g|^p w_{B,10E}\Big)^{\frac{1}{p}}\Big]^{\frac{1}{n}}\\
&=\Big[\prod_{i=1}^{n}\|E_{Q_i}g\|_{L^p(w_{B,10E})}\Big]^{\frac{1}{n}}\\
&\overset{(2.4)}{\le}\Big[\prod_{i=1}^{n}\delta^{-\frac{n-1}{4}}\Big(\sum_{q_i\in\mathrm{Part}_{\delta^{\frac{1}{2}}}(Q_i)}\|E_{q_i}g\|^2_{L^p(w_{B,10E})}\Big)^{\frac{1}{2}}\Big]^{\frac{1}{n}}\\
&=\delta^{-\frac{n-1}{4}}\Big[\prod_{i=1}^{n}\sum_{q_i\in\mathrm{Part}_{\delta^{\frac{1}{2}}}(Q_i)}\|E_{q_i}g\|^2_{L^p(w_{B,10E})}\Big]^{\frac{1}{2n}}\\
&\le\delta^{-\frac{n-1}{4}}\Big[\prod_{i=1}^{n}\sum_{q_i\in\mathrm{Part}_{\delta^{\frac{1}{2}}}(Q_i)}\|E_{q_i}g\|^2_{L^p(w_{B,E})}\Big]^{\frac{1}{2n}},
\end{aligned}$$

where the calculation of (2.4) was applied n times with $[0,1]^{n-1}$ replaced by Q_i, $1 \le i \le n$.

Similarly as in Lemma 2.6.1 we see that the constant $\mathrm{Dec}_n(\delta, p, \nu, m, E)$ is the infimum of the non-negative constants a satisfying the inequality. In particular,

$$\mathrm{Dec}_n(\delta, p, \nu, m, E) \le \delta^{-\frac{n-1}{4}}. \tag{6.1}$$

The following proposition gives us an estimate for the decoupling constant $\mathrm{Dec}_n(\delta, p, \nu, m, E)$ for all δ, when the decoupling constant can be estimated for a sparse, but not too sparse a subset of $4^{-\mathbb{N}}$. This estimate will be used in the final proof in Chapter 8.

Proposition 6.1.4. *Let $m \geq 2$. Put $\mathbb{N}_m := \{2^{m-1}k \colon k \in \mathbb{N}\}$. Let $\delta \in 4^{-\mathbb{N}}$ be such that $\delta \leq 4^{-2^m}$. Then there is a $\delta_1 \in 4^{-\mathbb{N}_m}$ such that $\delta \leq \delta_1$ and*

$$\mathrm{Dec}_n(\delta, p, \nu, m, E) \lesssim_{m,E,p,n} \mathrm{Dec}_n(\delta_1, p, \nu, m-1, E).$$

Proof. Let $n \geq 2$, $E > n$, $p \geq 2$, $m \in \mathbb{N}$ such that $m \geq 2$, $0 < \nu < 1$ and $\delta \in 4^{-\mathbb{N}}$ such that $\delta \leq 4^{-2^m}$. Let $\mu \geq \delta^{2^{-m}}$. Let $B \subset \mathbb{R}^n$ be a cube with $l(B) = \delta^{-1}$, $g \in L^1([0,1]^{n-1})$ and $Q_1, \ldots, Q_n$ ν-transverse cubes with equal side lengths μ.

Denote $C_m := 4^{2^{m-1}}$. Then there is a $\delta_0 \in 4^{-\mathbb{N}_m}$ such that $\delta_0 \leq \delta \leq C_m\delta_0 \in 4^{-\mathbb{N}_m}$. Put $\delta_1 := C_m\delta_0$. Observe that since $\delta \leq C_m^{-2}$,

$$\delta_1^{2^{-(m-1)}} = (C_m\delta_0)^{2^{-(m-1)}} = (C_m^2\delta_0^2)^{2^{-m}} \leq (C_m^2\delta^2)^{2^{-m}} = (C_m^2\delta\delta)^{2^{-m}} \leq \delta^{2^{-m}}.$$

Hence $\mu \geq \delta_1^{2^{-(m-1)}}$ and

$$\begin{aligned}
&\Big[\sum_{\Delta \in \mathrm{Part}_{\mu^{-1}}(B)} \Big(\prod_{i=1}^n \|E_{Q_i} g\|_{L^p(w_{\Delta,10E})}^p\Big)^{\frac{1}{n}}\Big]^{\frac{1}{p}} \\
&= \Big[\sum_{B_1 \in \mathrm{Part}_{\delta_1^{-1}}(B)} \sum_{\Delta \in \mathrm{Part}_{\mu^{-1}}(B_1)} \Big(\prod_{i=1}^n \|E_{Q_i} g\|_{L^p(w_{\Delta,10E})}^p\Big)^{\frac{1}{n}}\Big]^{\frac{1}{p}} \\
&\leq \Big[\sum_{B_1 \in \mathrm{Part}_{\delta_1^{-1}}(B)} \mathrm{Dec}_n(\delta_1, p, \nu, m-1, E)^p \Big[\prod_{i=1}^n \sum_{q_i \in \mathrm{Part}_{\delta_1^{\frac{1}{2}}}(Q_i)} \|E_{q_i} g\|_{L^p(w_{B_1,E})}^2\Big]^{\frac{p}{2n}}\Big]^{\frac{1}{p}} \\
&= \mathrm{Dec}_n(\delta_1, p, \nu, m-1, E)\Big[\sum_{B_1 \in \mathrm{Part}_{\delta_1^{-1}}(B)} \Big[\prod_{i=1}^n \sum_{q_i \in \mathrm{Part}_{\delta_1^{\frac{1}{2}}}(Q_i)} \|E_{q_i} g\|_{L^p(w_{B_1,E})}^2\Big]^{\frac{p}{2n}}\Big]^{\frac{1}{p}}.
\end{aligned}$$

Because $l(B_1) = \delta_1^{-1} \leq \delta^{-1} = l(B)$ and additionally $|c_B - c_{B_1}| \leq \frac{\sqrt{n}}{2}\delta^{-1}$, Lemma 4.1.6 implies that $w_{B_1,E} \lesssim_{E,n} w_{B,E}$. Thus continuing from the latest estimate we get

$$\begin{aligned}
&\Big[\sum_{B_1\in \mathrm{Part}_{\delta_1^{-1}}(B)}\Big[\prod_{i=1}^n \sum_{q_i\in \mathrm{Part}_{\delta_1^{\frac12}}(Q_i)} \|E_{q_i}g\|^2_{L^p(w_{B_1,E})}\Big]^{\frac{p}{2n}}\Big]^{\frac1p}\\
&\lesssim_{E,p,n} \Big[\sum_{B_1\in \mathrm{Part}_{\delta_1^{-1}}(B)}\Big[\prod_{i=1}^n \sum_{q_i\in \mathrm{Part}_{\delta_1^{\frac12}}(Q_i)} \|E_{q_i}g\|^2_{L^p(w_{B,E})}\Big]^{\frac{p}{2n}}\Big]^{\frac1p}\\
&= \Big[\Big(\frac{C_m\delta_0}{\delta}\Big)^n\Big[\prod_{i=1}^n \sum_{q_i\in \mathrm{Part}_{\delta_1^{\frac12}}(Q_i)} \|E_{q_i}g\|^2_{L^p(w_{B,E})}\Big]^{\frac{p}{2n}}\Big]^{\frac1p}\\
&\le \Big[C_m^n\Big[\prod_{i=1}^n \sum_{q_i\in \mathrm{Part}_{\delta_1^{\frac12}}(Q_i)} \|E_{q_i}g\|^2_{L^p(w_{B,E})}\Big]^{\frac{p}{2n}}\Big]^{\frac1p}\\
&\lesssim_{m,p,n} \Big[\prod_{i=1}^n \sum_{q_i\in \mathrm{Part}_{\delta_1^{\frac12}}(Q_i)} \|E_{q_i}g\|^2_{L^p(w_{B,E})}\Big]^{\frac{1}{2n}}
\end{aligned}$$

By essentially the same application of Minkowski's and Cauchy-Schwarz as in (2.4) we get for each $q_i\in \mathrm{Part}_{\delta_1^{\frac12}}(Q_i)$ that

$$\begin{aligned}
\|E_{q_i}g\|_{L^p(w_{B,E})} &= \|\sum_{r_i\in \mathrm{Part}_{\delta^{\frac12}}(q_i)} E_{r_i}g\|_{L^p(w_{B,E})}\\
&\le \Big(\sum_{r_i\in \mathrm{Part}_{\delta^{\frac12}}(q_i)} 1\Big)^{\frac12}\Big(\sum_{r_i\in \mathrm{Part}_{\delta^{\frac12}}(q_i)} \|E_{r_i}g\|^2_{L^p(w_{B,E})}\Big)^{\frac12}\\
&= \Big(\frac{C_m\delta_0}{\delta}\Big)^{\frac{n-1}{4}}\Big(\sum_{r_i\in \mathrm{Part}_{\delta^{\frac12}}(q_i)} \|E_{r_i}g\|^2_{L^p(w_{B,E})}\Big)^{\frac12}\\
&\le C_m^{\frac{n-1}{4}}\Big(\sum_{r_i\in \mathrm{Part}_{\delta^{\frac12}}(q_i)} \|E_{r_i}g\|^2_{L^p(w_{B,E})}\Big)^{\frac12}\\
&\lesssim_{m,n} \Big(\sum_{r_i\in \mathrm{Part}_{\delta^{\frac12}}(q_i)} \|E_{r_i}g\|^2_{L^p(w_{B,E})}\Big)^{\frac12}.
\end{aligned}$$

Plugging this into the preceding inequality we get

$$\Big[\prod_{i=1}^{n}\sum_{q_i\in \mathrm{Part}_{\delta_1^{\frac{1}{2}}}(Q_i)}\|E_{q_i}g\|^2_{L^p(w_{B,E})}\Big]^{\frac{1}{2n}}$$

$$\lesssim_{m,n}\Big[\prod_{i=1}^{n}\sum_{q_i\in \mathrm{Part}_{\delta_1^{\frac{1}{2}}}(Q_i)}\sum_{r_i\in \mathrm{Part}_{\delta^{\frac{1}{2}}}(q_i)}\|E_{r_i}g\|^2_{L^p(w_{B,E})}\Big]^{\frac{1}{2n}}$$

$$=\Big[\prod_{i=1}^{n}\sum_{q_i\in \mathrm{Part}_{\delta^{\frac{1}{2}}}(Q_i)}\|E_{q_i}g\|^2_{L^p(w_{B,E})}\Big]^{\frac{1}{2n}}.$$

Finally, combining the previous estimates we have

$$\Big[\sum_{\Delta\in \mathrm{Part}_{\mu^{-1}}(B)}\Big(\prod_{i=1}^{n}\|E_{Q_i}g\|^p_{L^p(w_{\Delta,10E})}\Big)^{\frac{1}{n}}\Big]^{\frac{1}{p}}$$

$$\lesssim_{m,E,p,n}\mathrm{Dec}_n(\delta_1,p,\nu,m-1,E)\Big[\prod_{i=1}^{n}\sum_{q_i\in \mathrm{Part}_{\delta^{\frac{1}{2}}}(Q_i)}\|E_{q_i}g\|^2_{L^p(w_{B,E})}\Big]^{\frac{1}{2n}},$$

which gives the result of this proposition. □

6.2 Induction and a multilinear Kakeya inequality

We present the following theorems without proofs. The proofs can be found in [3]. In [3], the first theorem is Theorem 8.1 and the second theorem is Theorem 9.2.

The first theorem forms the backbone of the induction to be used in the proof of Theorem 2.5.3. The condition (ii) shows that we need knowledge of the decoupling constant for possibly very large E.

Theorem 6.2.1. *For each $n \geq 2$ and $E \geq 100n$ there is a constant $\Gamma_n(E) \geq 100n$ such that the following statement holds.*

Fix $n \geq 2$, $E \geq 100n$ and $p \geq 2$. Assume that one of the following holds:

(i) $n = 2$.

(ii) $n \geq 3$ *and* $\mathrm{Dec}_{n-1}(\delta, p, \Gamma_{n-1}(10E)) \lesssim_{\epsilon,E,p,n} \delta^{-\epsilon}$ *holds for each* $\delta \in 4^{-\mathbb{N}}$ *and* $\epsilon > 0$.

Then there exists a function $\theta : (0,1) \to (0,\infty)$ with $\lim_{\nu\to 0}\theta(\nu) = 0$ such that for each $0 < \nu < 1$ and $m \in \mathbb{N}_+$ there are constants $C_{\nu,m} > 0$ and $D_{\nu,m} > 0$ such that

$$\mathrm{Dec}_n(R^{-1}, p, E) \le C_{\nu,m} R^{\theta(\nu)} \Big(1 + \sup_{1\le R' \le R} \mathrm{Dec}_n((R')^{-1}, p, \nu, m, E)\Big)$$

for each $R \in 4^{\mathbb{N}}$ such that $R \ge D_{\nu,m}$.

Here is another theorem that will be used. It is a consequence of a certain multilinear Kakeya inequality, as seen in [3].

Theorem 6.2.2. *Fix $E \ge 100n$. Let $p \ge \frac{2n}{n-1}$ and $0 < \delta < 1$, $\delta \in 2^{-\mathbb{N}}$ and $\epsilon \in (0,\infty)$. Let $0 < \nu < 1$. Consider n ν-transverse cubes $Q_1, \dots, Q_n \subset [0,1]^{n-1}$. Let B be a cube in $\mathbb{R}^n$ with $l(B) = \delta^{-2}$, and let $\mathcal{B}$ be the unique partition of B into cubes Δ with $l(\Delta) = \delta^{-1}$. Then for each $g \in L^1([0,1]^{n-1})$ we have*

$$\frac{1}{|\mathcal{B}|}\sum_{\Delta\in\mathcal{B}}\Big[\prod_{i=1}^{n}\Big(\sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \|E_{Q_{i,1}}g\|^2_{L^{\frac{p(n-1)}{n}}_{\#}(w_\Delta)}\Big)^{\frac12}\Big]^{\frac{p}{n}}$$

$$\lesssim_{\epsilon,\nu,E,p,n}\ \delta^{-\epsilon}\prod_{i=1}^{n}\Big(\sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \|E_{Q_{i,1}}g\|^2_{L^{\frac{p(n-1)}{n}}_{\#}(w_B)}\Big)^{\frac12}\Big]^{\frac{p}{n}}.$$

Chapter 7

The Iteration Scheme

In this chapter, we create a series of propositions, based on Proposition 5.1.1, whose purpose is to pave the way for the domination of $\mathrm{Dec}_n(\delta, p, \nu, m, E)$ with a power of δ in the final chapter. The final proposition, Proposition 7.3.2, is achieved by iterating the proposition that precedes it, hence the name of this chapter.

Let A be a set. If $A \subset \mathbb{R}^n$ and A has positive Lebesgue measure, then $|A|$ will refer to the Lebesgue measure of A. Otherwise, $|A|$ will refer to the cardinality of A.

7.1 Definitions

- In each proposition, lemma, corollary and remark of Chapter 7 and in these definitions, we assume the following:

  Let $n \geq 2$, $E \geq 100n$, $0 < \nu < 1$ and $0 < \delta < 1$ such that $\delta \in 2^{-\mathbb{N}}$. Additionally, let $Q_1, \ldots, Q_n \subset [0,1]^{n-1}$ be ν-transverse cubes with $\delta \leq l(Q_i) \leq 1$ for each i.

- For a positive integer $s \geq 1$, B^s will refer to cubes in $\mathbb{R}^n$ with side length $l(B^s) = \delta^{-s}$ and arbitrary centers.

- Let $t, p \geq 1$ and consider positive integers $1 \leq q \leq s \leq r$.

  For each cube B^r and $g \in L^1([0,1]^{n-1})$, we define

$$D_t(q, B^r, g) = \Big[\prod_{i=1}^{n}\Big(\sum_{Q_{i,q} \in \mathrm{Part}_{\delta^q}(Q_i)} \|E_{Q_{i,q}} g\|^2_{L^t_{\#}(w_{B^r})}\Big)^{\frac{1}{2}}\Big]^{\frac{1}{n}}.$$

  Here w_{B^r} is a shorthand for $w_{B^r,E}$.

  We define $\mathcal{B}_s(B^r) := \mathrm{Part}_{\delta^{-s}}(B^r)$. That is, $\mathcal{B}_s(B^r)$ is the unique essential partition of B^r with cubes B^s of side length δ^{-s}. Also, one can check that this unique partition always exists ($s \leq r$); thanks to our '$2^{\mathbb{Z}}$-assumption' concerning δ.

For each cube B^r and $g \in L^1([0,1]^{n-1})$, we define

$$A_p(q, B^r, s, g) = \Big(\frac{1}{|\mathcal{B}_s(B^r)|} \sum_{B^s \in \mathcal{B}_s(B^r)} D_2(q, B^s, g)^p\Big)^{\frac{1}{p}}.$$

Observe that if $r = s$, then $\mathcal{B}_r(B^r) = \{B^r\}$ and

$$A_p(q, B^r, r, g) = D_2(q, B^r, g). \tag{7.1}$$

- Fix $n \geq 2$. If $p > \frac{2n}{n-1}$, set

$$\kappa_p := \frac{pn - p - 2n}{(p-2)(n-1)}. \tag{7.2}$$

  If $2 \leq p \leq \frac{2n}{n-1}$, set $\kappa_p := 0$. Note: both p and n need to be known to define κ_p.

  Observe that the quotient in (7.2) is zero if $p = \frac{2n}{n-1}$. Also, $0 < \kappa_p < 1$ for all $p > \frac{2n}{n-1}$.

- When $\kappa_p = 0$, we make an agreement that $x^{\kappa_p} = 1$ for all $x \geq 0$.

7.2 Preparation for iteration

We record the following lemma for its usefulness in the upcoming propositions.

Lemma 7.2.1. *Let $p \geq 2$. Let $1 \leq q \leq r$ be positive integers. Then $D_2 \lesssim_{E,p,n} D_p$, that is,*

$$D_2(q, B^r, g) \lesssim_{E,p,n} D_p(q, B^r, g)$$

for all cubes B^r and $g \in L^1([0,1]^{n-1})$.

Proof. Clearly we can assume that $p > 2$. Fix integers $r \geq q \geq 1$. Fix B^r. Assume that $g \in L^1([0,1]^{n-1})$.

Let $1 \le i \le n$. Let $Q_{i,q} \in \mathrm{Part}_{\delta^q}(Q_i)$. We denote $h := E_{Q_{i,q}} g$. Now $\alpha := \frac{p}{2} > 1$. By Hölder's inequality

$$\begin{aligned}
\|h\|_{L^2_{\#}(w_{B^r})} &= \Big(\frac{1}{|B^r|}\int |h|^2 w_{B^r}\Big)^{\frac{1}{2}} \\
&= \Big(\frac{1}{|B^r|}\int |h|^2 w_{B^r}^{\frac{1}{\alpha}} w_{B^r}^{\frac{\alpha-1}{\alpha}}\Big)^{\frac{1}{2}} \\
&\le \Big[\frac{1}{|B^r|}\Big(\int |h|^{2\alpha} w_{B^r}\Big)^{\frac{1}{\alpha}}\Big(\int w_{B^r}\Big)^{\frac{\alpha-1}{\alpha}}\Big]^{\frac{1}{2}} \\
&\sim_{E,p,n} \big(|B^r|\big)^{-\frac{1}{2}}\Big(\int |h|^p w_{B^r}\Big)^{\frac{1}{p}}\big(|B^r|\big)^{\frac{p-2}{2p}} \\
&= \Big(\frac{1}{|B^r|}\int |h|^p w_{B^r}\Big)^{\frac{1}{p}} \\
&= \|h\|_{L^p_{\#}(w_{B^r})}.
\end{aligned}$$

Hence we get for all $p > 2$ that

$$\|E_{Q_{i,q}} g\|_{L^2_{\#}(w_{B^r})} \lesssim_{E,p,n} \|E_{Q_{i,q}} g\|_{L^p_{\#}(w_{B^r})}$$

for all $1 \le i \le n$ and $Q_{i,q} \in \mathrm{Part}_{\delta^q}(Q_i)$. After applying this to the definitions of D_2 and D_p, the claim is proved. $\square$

Proposition 7.2.2. *We have for each cube B^2, $p \ge 2$, $g \in L^1([0,1]^{n-1})$ and $\epsilon > 0$,*

$$A_p(1, B^2, 1, g) \lesssim_{\epsilon,\nu,E,p,n} \delta^{-\epsilon} A_p(2, B^2, 2, g)^{1-\kappa_p} D_p(1, B^2, g)^{\kappa_p}.$$

Proof. Assume that $g \in L^1([0,1]^{n-1})$. Let B^2 be fixed. Let $\epsilon > 0$ and let $p \ge 2$.

We assume for the first part of the proof that $p \ge \frac{2n}{n-1}$. Now $2 \le \frac{p(n-1)}{n}$. By using Lemma 7.2.1 we get

$$\begin{aligned}
A_p(1, B^2, 1, g)^p &= \frac{1}{|\mathcal{B}_1(B^2)|}\sum_{B^1 \in \mathcal{B}_1(B^2)} D_2(1, B^1, g)^p \\
&\lesssim_{E,p,n} \frac{1}{|\mathcal{B}_1(B^2)|}\sum_{B^1 \in \mathcal{B}_1(B^2)} D_{\frac{p(n-1)}{n}}(1, B^1, g)^p \\
&= \frac{1}{|\mathcal{B}_1(B^2)|}\sum_{B^1 \in \mathcal{B}_1(B^2)} \Big[\prod_{i=1}^{n}\Big(\sum_{Q_{i,1} \in \mathrm{Part}_{\delta}(Q_i)} \|E_{Q_{i,1}} g\|^2_{L_{\#}^{\frac{p(n-1)}{n}}(w_{B^1})}\Big)^{\frac{1}{2}}\Big]^{\frac{p}{n}},
\end{aligned}$$

since the implicit constant does not depend on anything else except possibly on E, p or n. Now by Theorem 6.2.2 we estimate the term above with

$$\begin{aligned}
&\lesssim_{\epsilon,\nu,E,p,n} \delta^{-\epsilon}\Big[\prod_{i=1}^{n}\Big(\sum_{Q_{i,1} \in \mathrm{Part}_{\delta}(Q_i)} \|E_{Q_{i,1}} g\|^2_{L_{\#}^{\frac{p(n-1)}{n}}(w_{B^2})}\Big)^{\frac{1}{2}}\Big]^{\frac{p}{n}} \\
&\le \delta^{-\epsilon p}\Big[\Big(\prod_{i=1}^{n}\sum_{Q_{i,1} \in \mathrm{Part}_{\delta}(Q_i)} \|E_{Q_{i,1}} g\|^2_{L_{\#}^{\frac{p(n-1)}{n}}(w_{B^2})}\Big)^{\frac{1}{2n}}\Big]^{p}
\end{aligned}$$

Hence

$$A_p(1,B^2,1,g) \lesssim_{\epsilon,\nu,E,p,n} \delta^{-\epsilon}\Big(\prod_{i=1}^n \sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \|E_{Q_{i,1}}g\|^2_{L_\#^{\frac{p(n-1)}{n}}(w_{B^2})}\Big)^{\frac{1}{2n}}. \tag{7.3}$$

On the right,

$$\begin{aligned}
&\prod_{i=1}^n \sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \|E_{Q_{i,1}}g\|^2_{L_\#^{\frac{p(n-1)}{n}}(w_{B^2})}\\
&=\prod_{i=1}^n \sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \Big(\frac{1}{|B^2|}\int |E_{Q_{i,1}}g|^{\frac{p(n-1)}{n}} w_{B^2}\Big)^{\frac{2n}{p(n-1)}}\\
&=\prod_{i=1}^n \Big(\frac{1}{|B^2|}\Big)^{\frac{2n}{p(n-1)}} \sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \Big(\int |E_{Q_{i,1}}g|^{\frac{p(n-1)}{n}} w_{B^2}\Big)^{\frac{2n}{p(n-1)}}. \qquad (7.4)
\end{aligned}$$

We proceed to handle the sum in (7.4). Assume first that $p > \frac{2n}{n-1}$. For all $1 \le i \le n$. By Hölder's inequality (first for integrals and then for sums)

$$\begin{aligned}
&\sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \Big(\int |E_{Q_{i,1}}g|^{\frac{p(n-1)}{n}} w_{B^2}\Big)^{\frac{2n}{p(n-1)}}\\
&= \sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \Big(\int |E_{Q_{i,1}}g|^{\frac{2p}{(p-2)n}} w_{B^2}^{\frac{p}{(p-2)n}} |E_{Q_{i,1}}g|^{\frac{p(pn-p-2n)}{(p-2)n}} w_{B^2}^{\frac{pn-p-2n}{(p-2)n}}\Big)^{\frac{2n}{p(n-1)}}\\
&\le \sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \Big[\Big(\int |E_{Q_{i,1}}g|^2 w_{B^2}\Big)^{\frac{p}{(p-2)n}} \Big(\int |E_{Q_{i,1}}g|^p w_{B^2}\Big)^{\frac{pn-p-2n}{(p-2)n}}\Big]^{\frac{2n}{p(n-1)}}\\
&= \sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \Big(\int |E_{Q_{i,1}}g|^2 w_{B^2}\Big)^{1-\kappa_p} \Big(\int |E_{Q_{i,1}}g|^p w_{B^2}\Big)^{\frac{2\kappa_p}{p}}\\
&\le \Big(\sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \int |E_{Q_{i,1}}g|^2 w_{B^2}\Big)^{1-\kappa_p} \Big(\sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \Big(\int |E_{Q_{i,1}}g|^p w_{B^2}\Big)^{\frac{2}{p}}\Big)^{\kappa_p}.
\end{aligned}$$

By (7.4) and the previous inequality

$$
\begin{aligned}
&\prod_{i=1}^{n} \sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \|E_{Q_{i,1}}g\|^2_{L_\#^{\frac{p(n-1)}{n}}(w_{B^2})} \\
&\le \prod_{i=1}^{n} \Big(\frac{1}{|B^2|}\Big)^{\frac{2n}{p(n-1)}} \Big(\sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \int |E_{Q_{i,1}}g|^2 w_{B^2}\Big)^{1-\kappa_p} \\
&\quad \cdot \Big(\sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \Big(\int |E_{Q_{i,1}}g|^p w_{B^2}\Big)^{\frac{2}{p}}\Big)^{\kappa_p} \\
&= \prod_{i=1}^{n} \Big(\sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \frac{1}{|B^2|}\int |E_{Q_{i,1}}g|^2 w_{B^2}\Big)^{1-\kappa_p} \\
&\quad \cdot \Big(\sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \Big(\frac{1}{|B^2|}\int |E_{Q_{i,1}}g|^p w_{B^2}\Big)^{\frac{2}{p}}\Big)^{\kappa_p} \\
&= \prod_{i=1}^{n} \Big(\sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \|E_{Q_{i,1}}g\|^2_{L^2_\#(w_{B^2})}\Big)^{1-\kappa_p} \Big(\sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \|E_{Q_{i,1}}g\|^2_{L^p_\#(w_{B^2})}\Big)^{\kappa_p} \\
&= \Big(\prod_{i=1}^{n}\sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \|E_{Q_{i,1}}g\|^2_{L^2_\#(w_{B^2})}\Big)^{1-\kappa_p} \Big(\prod_{i=1}^{n}\sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \|E_{Q_{i,1}}g\|^2_{L^p_\#(w_{B^2})}\Big)^{\kappa_p} \\
&= \Big(\prod_{i=1}^{n}\sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \|E_{Q_{i,1}}g\|^2_{L^2_\#(w_{B^2})}\Big)^{1-\kappa_p} D_p(1,B^2,g)^{2n\kappa_p} \qquad (7.5)
\end{aligned}
$$

If on the other hand $p = \frac{2n}{n-1}$, then $\frac{p(n-1)}{n} = 2$, $\kappa_p = 0$ and

$$
\begin{aligned}
&\prod_{i=1}^{n} \sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \|E_{Q_{i,1}}g\|^2_{L_\#^{\frac{p(n-1)}{n}}(w_{B^2})} \\
&= \Big(\prod_{i=1}^{n}\sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \|E_{Q_{i,1}}g\|^2_{L^2_\#(w_{B^2})}\Big)^{1-\kappa_p}. \qquad (7.6)
\end{aligned}
$$

By (7.3), (7.5) and (7.6) we have

$$
A_p(1,B^2,1,g) \lesssim_{\epsilon,\nu,E,p,n} \delta^{-\epsilon}\Big(\prod_{i=1}^{n}\sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \|E_{Q_{i,1}}g\|^2_{L^2_\#(w_{B^2})}\Big)^{\frac{1-\kappa_p}{2n}} D_p(1,B^2,g)^{\kappa_p}, \qquad (7.7)
$$

which is almost the claim of this proposition.

We proceed to handle the L^2-term in the above formula. Let $1 \le i \le n$. Let $Q_{i,1} \in \mathrm{Part}_\delta(Q_i)$. Since $0 < \delta < 1$, we know that $l(Q_{i,1}) = \delta > \delta^2 = (\delta^{-2})^{-1} = l(B^2)^{-1}$. Hence, by Proposition 5.1.1,

$$
\|E_{Q_{i,1}}g\|^2_{L^2(w_{B^2})} \lesssim_{E,n} \sum_{q\in \mathrm{Part}_{\delta^2}(Q_{i,1})} \|E_q g\|^2_{L^2(w_{B^2})}.
$$

Since the implicit constant above is the same for all $Q_{i,1}$, we can estimate

$$\begin{aligned}\sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \|E_{Q_{i,1}}g\|^2_{L^2_\#(w_{B^2})} &= \frac{1}{|B^2|}\sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \|E_{Q_{i,1}}g\|^2_{L^2(w_{B^2})} \\ &\lesssim_{E,n} \frac{1}{|B^2|}\sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \Big(\sum_{q\in \mathrm{Part}_{\delta^2}(Q_{i,1})} \|E_q g\|^2_{L^2(w_{B^2})}\Big) \\ &= \frac{1}{|B^2|}\sum_{Q_{i,2}\in \mathrm{Part}_{\delta^2}(Q_i)} \|E_{Q_{i,2}}g\|^2_{L^2(w_{B^2})} \\ &= \sum_{Q_{i,2}\in \mathrm{Part}_{\delta^2}(Q_i)} \|E_{Q_{i,2}}g\|^2_{L^2_\#(w_{B^2})}.\end{aligned}$$

Let again $p \geq \frac{2n}{n-1}$ be arbitrary. We get

$$\Big(\sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \|E_{Q_{i,1}}g\|^2_{L^2_\#(w_{B^2})}\Big)^{\frac{1-\kappa_p}{2n}} \lesssim_{E,p,n} \Big(\sum_{Q_{i,2}\in \mathrm{Part}_{\delta^2}(Q_i)} \|E_{Q_{i,2}}g\|^2_{L^2_\#(w_{B^2})}\Big)^{\frac{1-\kappa_p}{2n}}.$$

Since the implicit constant above is the same for all Q_i, we can take products and estimate

$$\begin{aligned}&\Big(\prod_{i=1}^n \sum_{Q_{i,1}\in \mathrm{Part}_\delta(Q_i)} \|E_{Q_{i,1}}g\|^2_{L^2_\#(w_{B^2})}\Big)^{\frac{1-\kappa_p}{2n}} \\ &\lesssim_{E,p,n} \Big(\prod_{i=1}^n \sum_{Q_{i,2}\in \mathrm{Part}_{\delta^2}(Q_i)} \|E_{Q_{i,2}}g\|^2_{L^2_\#(w_{B^2})}\Big)^{\frac{1-\kappa_p}{2n}} \\ &= D_2(2, B^2, g)^{1-\kappa_p} \\ &= A_p(2, B^2, 2, g)^{1-\kappa_p}.\end{aligned}$$

Thus by (7.7)

$$A_p(1, B^2, 1, g) \lesssim_{E,p,n} \delta^{-\epsilon} A_p(2, B^2, 2, g)^{1-\kappa_p} D_p(1, B^2, g)^{\kappa_p}.$$

This finishes the proof when $p \geq \frac{2n}{n-1}$.

Assume then that $2 \leq p < \frac{2n}{n-1}$. Then $\kappa_p = 0$. Now by averaged Hölder's inequality (3.1.1) and the first part of the proof we get

$$\begin{aligned}
A_p(1,B^2,1,g) &= \Big(\frac{1}{|\mathcal{B}_1(B^2)|}\sum_{B^1\in\mathcal{B}_1(B^2)} D_2(1,B^1,g)^p\Big)^{\frac{1}{p}}\\
&\le \Big(\frac{1}{|\mathcal{B}_1(B^2)|}\sum_{B^1\in\mathcal{B}_1(B^2)} D_2(1,B^1,g)^{\frac{2n}{n-1}}\Big)^{\frac{n-1}{2n}}\\
&= A_{\frac{2n}{n-1}}(1,B^2,1,g)\\
&\lesssim_{\epsilon,\nu,E,n} \delta^{-\epsilon} A_{\frac{2n}{n-1}}(2,B^2,2,g)\\
&= \delta^{-\epsilon} D_2(2,B^2,g)\\
&= \delta^{-\epsilon} A_p(2,B^2,2,g)^{1-\kappa_p},
\end{aligned}$$

where in the last lines we used (7.1). This completes the proof of Proposition 7.2.2. $\square$

Let us formulate another useful lemma.

Lemma 7.2.3. *Let $p \ge 2$. Let $1 \le q \le s \le r$ be positive integers. Then*

$$\frac{1}{|\mathcal{B}_s(B^r)|}\sum_{B^s\in\mathcal{B}_s(B^r)} D_p(q,B^s,g)^p \lesssim_{E,n} D_p(q,B^r,g)^p.$$

for all cubes B^r and $g \in L^1([0,1]^{n-1})$.

Proof. Fix $p \ge 2$. Fix integers $r \ge s \ge q \ge 1$. Fix B^r. Assume that $g \in$

$L^1([0,1]^{n-1})$. Then

$$
\begin{aligned}
&\frac{1}{|\mathcal{B}_s(B^r)|}\sum_{B^s\in\mathcal{B}_s(B^r)} D_p(q,B^s,g)^p\\
&=\frac{|B^s|}{|B^r|}\sum_{B^s\in\mathcal{B}_s(B^r)}\Big[\prod_{i=1}^n\Big(\sum_{Q_{i,q}\in\mathrm{Part}_{\delta^q}(Q_i)}\|E_{Q_{i,q}}g\|^2_{L^p_\#(w_{B^s})}\Big)^{\frac{p}{2}}\Big]^{\frac{1}{n}}\\
&=\sum_{B^s\in\mathcal{B}_s(B^r)}\prod_{i=1}^n\Big[\Big(\sum_{Q_{i,q}\in\mathrm{Part}_{\delta^q}(Q_i)}\Big(\frac{1}{|B^r|}\int|E_{Q_{i,q}}g|^p w_{B^s}\Big)^{\frac{2}{p}}\Big)^{\frac{p}{2}}\Big]^{\frac{1}{n}}\\
&\overset{(\mathrm{H})}{\le}\Big[\prod_{i=1}^n\sum_{B^s\in\mathcal{B}_s(B^r)}\Big(\sum_{Q_{i,q}\in\mathrm{Part}_{\delta^q}(Q_i)}\Big(\frac{1}{|B^r|}\int|E_{Q_{i,q}}g|^p w_{B^s}\Big)^{\frac{2}{p}}\Big)^{\frac{p}{2}}\Big]^{\frac{1}{n}}\\
&\overset{(\mathrm{r.M.})}{\le}\Big[\prod_{i=1}^n\Big(\sum_{Q_{i,q}\in\mathrm{Part}_{\delta^q}(Q_i)}\Big(\sum_{B^s\in\mathcal{B}_s(B^r)}\frac{1}{|B^r|}\int|E_{Q_{i,q}}g|^p w_{B^s}\Big)^{\frac{2}{p}}\Big)^{\frac{p}{2}}\Big]^{\frac{1}{n}}\\
&\overset{(\text{Lemma 4.1.5})}{\lesssim_{E,n}}\Big[\prod_{i=1}^n\Big(\sum_{Q_{i,q}\in\mathrm{Part}_{\delta^q}(Q_i)}\Big(\frac{1}{|B^r|}\int|E_{Q_{i,q}}g|^p w_{B^r}\Big)^{\frac{2}{p}}\Big)^{\frac{p}{2}}\Big]^{\frac{1}{n}}\\
&=\Big[\prod_{i=1}^n\Big(\sum_{Q_{i,q}\in\mathrm{Part}_{\delta^q}(Q_i)}\|E_{Q_{i,q}}g\|^2_{L^p_\#(w_{B^r})}\Big)^{\frac{1}{2}}\Big]^{\frac{p}{n}}\\
&=D_p(q,B^r,g)^p.
\end{aligned}
$$

In (H), we used the n-product Hölder's inequality. In (r.M.), we used the reverse Minkowski's inequality in $\frac{2}{p}$. ☐

Corollary 7.2.4. *Let $p\ge 2$. Let $1\le q\le s\le r$ be positive integers. Then*

$$A_p(q,B^r,s,g)\lesssim_{E,p,n} D_p(q,B^r,g)$$

for all cubes B^r and $g\in L^1([0,1]^{n-1})$.

Proof. By Lemma 7.2.1 and Lemma 7.2.3,

$$
\begin{aligned}
A_p(q,B^r,s,g)^p &= \frac{1}{|\mathcal{B}_s(B^r)|}\sum_{B^s\in\mathcal{B}_s(B^r)} D_2(q,B^s,g)^p\\
&\lesssim_{E,p,n}\frac{1}{|\mathcal{B}_s(B^r)|}\sum_{B^s\in\mathcal{B}_s(B^r)} D_p(q,B^s,g)^p\\
&\lesssim_{E,n} D_p(q,B^r,g)^p.
\end{aligned}
$$

☐

Remark 7.2.5. Let $p \geq 2$. Fix B^2 and $g \in L^1([0,1]^{n-1})$. Let $\epsilon > 0$. Observe that by Corollary 7.2.4

$$A_p(1, B^2, 1, g) \lesssim_{E,p,n} D_p(1, B^2, g) \leq \delta^{-\epsilon} D_p(1, B^2, g).$$

This shows that Proposition 7.2.2 is true with κ_p replaced by 1.

Remark 7.2.6. Furthermore, we can also consider exponents between κ_p and 1 by using Proposition 7.2.2 and the previous remark. Namely, the proposition and the remark imply that

$$\begin{cases} A_p \leq C_1 \delta^{-\epsilon} A_p^{1-\kappa_p} D_p^{\kappa_p} \\ A_p \leq C_2 \delta^{-\epsilon} D_p, \end{cases}$$

where C_1 depends on ϵ, ν, E, p and n and C_2 depends on E, p and n. Let $C = \max\{C_1, C_2\}$. Let $0 < \theta < 1$. Raising to powers θ and $1-\theta$ we get

$$\begin{cases} A_p^\theta \leq C^\theta \delta^{-\epsilon\theta} A_p^{(1-\kappa_p)\theta} D_p^{\kappa_p \theta} \\ A_p^{1-\theta} \leq C^{1-\theta} \delta^{-\epsilon(1-\theta)} D_p^{1-\theta} \end{cases}$$

Thus

$$A_p = A_p^\theta A_p^{1-\theta} \leq C \delta^{-\epsilon} A_p^{(1-\kappa_p)\theta} D_p^{1-(1-\kappa_p)\theta},$$

where C only depends on ϵ, ν, E, p and n. Note that $t := 1 - (1-\kappa_p)\theta$ takes all the values in $(\kappa_p, 1)$, when $0 < \theta < 1$. Thus for all $\kappa_p < t < 1$

$$A_p(1, B^2, 1, g) \lesssim_{\epsilon,\nu,E,p,n} \delta^{-\epsilon} A_p(2, B^2, 2, g)^{1-t} D_p(1, B^2, g)^t.$$

Lemma 7.2.7. *Let $p \geq 2$. Then $\kappa_p < \frac{1}{2}$ if and only if $p < \frac{2(n+1)}{n-1}$.*

Proof. If $2 \leq p \leq 2n/(n-1)$, then $\kappa_p = 0 < 1/2$ and $p < 2(n+1)/(n-1)$. Assume on the other hand that $p > 2n/(n-1)$. Now

$$\begin{aligned} \kappa_p < \tfrac{1}{2} &\Leftrightarrow \tfrac{pn-p-2n}{(p-2)(n-1)} < \tfrac{1}{2} \\ &\Leftrightarrow pn - p - 2n < \tfrac{1}{2}pn - \tfrac{1}{2}p - n + 1 \\ &\Leftrightarrow \tfrac{1}{2}pn - \tfrac{1}{2}p < n + 1 \\ &\Leftrightarrow p < \tfrac{2(n+1)}{n-1} \end{aligned}$$

□

Proposition 7.2.8. *We have for each cube B^M with $M \geq 2$, $p \geq 2$, integrable $g : [0,1]^{n-1} \to \mathbb{C}$ and $\epsilon > 0$ that*

$$A_p(1, B^M, 1, g) \lesssim_{\epsilon,\nu,E,p,n} \delta^{-\epsilon} A_p(2, B^M, 2, g)^{1-\kappa_p} D_p(1, B^M, g)^{\kappa_p}.$$

Especially, the implicit constant is independent of M.

Proof. Assume that $g \in L^1([0,1]^{n-1})$. Let $\epsilon > 0$. Let $M \geq 2$ be an integer and fix B^M. Let $p \geq 2$.

Observe that $|\mathcal{B}_1(B^2)|^{-1}$ does not depend on the cube $B^2 \in \mathcal{B}_2(B^M)$ and we get

$$\begin{aligned}
&\sum_{B^2 \in \mathcal{B}_2(B^M)} A_p(1, B^2, 1, g)^p \\
&= \sum_{B^2 \in \mathcal{B}_2(B^M)} \frac{1}{|\mathcal{B}_1(B^2)|} \sum_{B^1 \in \mathcal{B}_1(B^2)} D_2(1, B^1, g)^p \\
&= \frac{1}{|\mathcal{B}_1(B^2)|} \sum_{B^2 \in \mathcal{B}_2(B^M)} \sum_{B^1 \in \mathcal{B}_1(B^2)} D_2(1, B^1, g)^p \\
&= \frac{1}{|\mathcal{B}_1(B^2)|} \sum_{B^1 \in \mathcal{B}_1(B^M)} D_2(1, B^1, g)^p \\
&= \frac{|\mathcal{B}_1(B^M)|}{|\mathcal{B}_1(B^2)|} A_p(1, B^M, 1, g)^p \\
&= \delta^{(2-M)n} A_p(1, B^M, 1, g)^p,
\end{aligned}$$

that is,

$$A_p(1, B^M, 1, g)^p = \delta^{(M-2)n} \sum_{B^2 \in \mathcal{B}_2(B^M)} A_p(1, B^2, 1, g)^p. \tag{7.8}$$

We first consider $p > \frac{2n}{n-1}$. Let $B^2 \in \mathcal{B}_2(B^M)$. Let us raise the claim of Proposition 7.2.2 to the power p and we get (recalling (7.1)) that

$$\begin{aligned}
A_p(1, B^2, 1, g)^p &= \frac{1}{|\mathcal{B}_1(B^2)|} \sum_{B^1 \in \mathcal{B}_1(B^2)} D_2(1, B^1, g)^p \\
&\lesssim_{\epsilon,\nu,E,p,n} \delta^{-\epsilon p} D_2(2, B^2, g)^{(1-\kappa_p)p} D_p(1, B^2, g)^{\kappa_p p}
\end{aligned}$$

By summing over $B^2 \in \mathcal{B}_2(B^M)$ we get (the implicit constant does not depend on B^2)

$$\begin{aligned}
&\sum_{B^2 \in \mathcal{B}_2(B^M)} A_p(1, B^2, 1, g)^p \\
&\lesssim_{\epsilon,\nu,E,p,n} \sum_{B^2 \in \mathcal{B}_2(B^M)} \delta^{-\epsilon p} D_2(2, B^2, g)^{(1-\kappa_p)p} D_p(1, B^2, g)^{\kappa_p p} \\
&= \delta^{-\epsilon p} \sum_{B^2 \in \mathcal{B}_2(B^M)} D_2(2, B^2, g)^{(1-\kappa_p)p} D_p(1, B^2, g)^{\kappa_p p} \\
&\leq \delta^{-\epsilon p} \Big(\sum_{B^2 \in \mathcal{B}_2(B^M)} D_2(2, B^2, g)^p \Big)^{1-\kappa_p} \Big(\sum_{B^2 \in \mathcal{B}_2(B^M)} D_p(1, B^2, g)^p \Big)^{\kappa_p} \qquad \text{(H)} \\
&= \delta^{-\epsilon p} \Big(|\mathcal{B}_2(B^M)| A_p(2, B^M, 2, g)^p \Big)^{1-\kappa_p} \Big(\sum_{B^2 \in \mathcal{B}_2(B^M)} D_p(1, B^2, g)^p \Big)^{\kappa_p}
\end{aligned}$$

In (H), we used Hölder's inequality in the spaces $l^{\frac{1}{1-\kappa_p}}(\mathcal{B}_2(B^M)), l^{\frac{1}{\kappa_p}}(\mathcal{B}_2(B^M))$ where $\frac{1}{1-\kappa_p}, \frac{1}{\kappa_p} \in (1,\infty)$.

Hence recalling (7.8) we get

$$\begin{aligned}
&A_p(1,B^M,1,g)^p \\
&\lesssim_{\epsilon,\nu,E,p,n} \delta^{(M-2)n}\delta^{-\epsilon p}\Big(|\mathcal{B}_2(B^M)|A_p(2,B^M,2,g)^p\Big)^{1-\kappa_p}\Big(\sum_{B^2\in\mathcal{B}_2(B^M)} D_p(1,B^2,g)^p\Big)^{\kappa_p} \\
&= \delta^{-\epsilon p}A_p(2,B^M,2,g)^{p(1-\kappa_p)}\delta^{n\kappa_p(M-2)}\Big(\sum_{B^2\in\mathcal{B}_2(B^M)} D_p(1,B^2,g)^p\Big)^{\kappa_p},
\end{aligned}$$

which is almost the claim of this proposition.

It now suffices to show that

$$\delta^{n(M-2)}\cdot\sum_{B^2\in\mathcal{B}_2(B^M)} D_p(1,B^2,g)^p \lesssim_{E,n} D_p(1,B^M,g)^p.$$

But this is equivalent with

$$\frac{1}{|\mathcal{B}_2(B^M)|}\sum_{B^2\in\mathcal{B}_2(B^M)} D_p(1,B^2,g)^p \lesssim_{E,n} D_p(1,B^M,g)^p,$$

which follows from Lemma 7.2.3.

We then consider $2 \le p \le \frac{2n}{n-1}$. Now $\kappa_p = 0$. Then (again) by Proposition 7.2.2 and by summing over $B^2 \in \mathcal{B}_2(B^M)$ we get (the implicit constant does not depend on B^2)

$$\begin{aligned}
A_p(1,B^M,1,g)^p &\overset{(7.8)}{=} \frac{|\mathcal{B}_1(B^2)|}{|\mathcal{B}_1(B^M)|}\sum_{B^2\in\mathcal{B}_2(B^M)} A_p(1,B^2,1,g)^p \\
&\lesssim_{\epsilon,\nu,E,p,n} \frac{|\mathcal{B}_1(B^2)|}{|\mathcal{B}_1(B^M)|}\sum_{B^2\in\mathcal{B}_2(B^M)} \delta^{-\epsilon p}A_p(2,B^2,2,g)^p \\
&\overset{(7.1)}{=} \delta^{(M-2)n}\sum_{B^2\in\mathcal{B}_2(B^M)} \delta^{-\epsilon p}D_2(2,B^2,g)^p \\
&= \delta^{-\epsilon p}\frac{1}{|\mathcal{B}_2(B^M)|}\sum_{B^2\in\mathcal{B}_2(B^M)} D_2(2,B^2,g)^p \\
&= \delta^{-\epsilon p}\frac{1}{|\mathcal{B}_2(B^M)|}|\mathcal{B}_2(B^M)|A_p(2,B^M,2,g)^p \\
&= \delta^{-\epsilon p}A_p(2,B^M,2,g)^{(1-\kappa_p)p},
\end{aligned}$$

which proves the claim. $\square$

7.3 Iteration

Proposition 7.3.1 is iterated to obtain Proposition 7.3.2.

Proposition 7.3.1. *Let $l, m \in \mathbb{N}$ with $l+1 \le m$. We have for each cube B^{2^m}, $p \ge 2$, $g \in L^1([0,1]^{n-1})$ and $\epsilon > 0$,*

$$A_p(2^l, B^{2^m}, 2^l, g) \lesssim_{\epsilon,\nu,E,p,n} \delta^{-2^l\epsilon} A_p(2^{l+1}, B^{2^m}, 2^{l+1}, g)^{1-\kappa_p} D_p(2^l, B^{2^m}, g)^{\kappa_p}.$$

The implicit constant is independent of l and m.

Proof. Let $l, m \in \mathbb{N}$ with $l+1 \le m$. Assume that $g \in L^1([0,1]^{n-1})$ and fix B^{2^m}. Let $p \ge 2$ and let $\epsilon > 0$.

We aim to use Proposition 7.2.8. Because $m \ge l+1$, we deduce that $2^{m-l} \ge 2$. Since $0 < \delta < 1$ we know that $0 < \delta^{2^l} \le \delta < 1$. Also, $\delta^{2^l} \in 2^{-\mathbb{N}}$.

Denote $\delta' := \delta^{2^l}$. Observe that $l(B^{2^m}) = \delta^{-2^m} = (\delta')^{-2^{m-l}}$. Also, $l(Q_i) \ge \delta \ge \delta'$ for all $1 \le i \le n$. In what follows, all quantities D'_t and A'_t as well as cubes $(B^s)'$ and partitions $\mathcal{B}'_s$ are relative to δ', not δ. Using Proposition 7.2.8 with $M = 2^{m-l}$, and δ' instead of δ, we get

$$A'_p(1, (B^{2^{m-l}})', 1, g) \lesssim_{\epsilon,\nu,E,p,n} (\delta')^{-\epsilon} A'_p(2, (B^{2^{m-l}})', 2, g)^{1-\kappa_p} D'_p(1, (B^{2^{m-l}})', g)^{\kappa_p}.$$

By opening up the expressions this is seen to be the same as

$$\begin{aligned}
&\Big(\frac{1}{|\mathcal{B}_{2^l}(B^{2^m})|} \sum_{B^{2^l} \in \mathcal{B}_{2^l}(B^{2^m})} D_2(2^l, B^{2^l}, g)^p\Big)^{\frac{1}{p}} \\
&\lesssim_{\epsilon,\nu,E,p,n} (\delta^{2^l})^{-\epsilon} \Big(\frac{1}{|\mathcal{B}_{2^{l+1}}(B^{2^m})|} \sum_{B^{2^{l+1}} \in \mathcal{B}_{2^{l+1}}(B^{2^m})} D_2(2^{l+1}, B^{2^{l+1}}, g)^p\Big)^{\frac{1-\kappa_p}{p}} \\
&\qquad \cdot \Big[\prod_{i=1}^n \Big(\sum_{Q_{i,2^l} \in \mathrm{Part}_{\delta^{2^l}}(Q_i)} \|E_{Q_{i,2^l}} g\|^2_{L^p_\#(w_{B^{2^m}})}\Big)^{\frac{1}{2}}\Big]^{\frac{\kappa_p}{n}}
\end{aligned}$$

which in turn is the same as

$$A_p(2^l, B^{2^m}, 2^l, g) \lesssim_{\epsilon,\nu,E,p,n} \delta^{-2^l\epsilon} A_p(2^{l+1}, B^{2^m}, 2^{l+1}, g)^{1-\kappa_p} D_p(2^l, B^{2^m}, g)^{\kappa_p}.$$

□

Then we iterate Proposition 7.3.1.

Proposition 7.3.2. *Let $m \in \mathbb{N}_+$. We have for each cube B^{2^m}, $p \ge 2$, $g \in L^1([0,1]^{n-1})$ and $\epsilon > 0$,*

$$\begin{aligned}
&A_p(1, B^{2^m}, 1, g) \\
&\lesssim_{\epsilon,\nu,m,E,p,n} \delta^{-\epsilon} A_p(2^{m-1}, B^{2^m}, 2^{m-1}, g)^{(1-\kappa_p)^{m-1}} \prod_{l=0}^{m-2} D_p(2^l, B^{2^m}, g)^{\kappa_p(1-\kappa_p)^l}.
\end{aligned}$$

Proof. Let $m \in \mathbb{N}, m \geq 1$ and $p \geq 2$. Fix g and $\epsilon > 0$. Let $B^{2^m} \subset \mathbb{R}^n$ be a cube.

Assume first that $m = 1$. Then because $1 < \delta^{-\epsilon}$ we can write

$$A_p(1, B^2, 1, g) \leq \delta^{-\epsilon} A_p(1, B^2, 1, g)^{(1-\kappa_p)^0}$$

and hence the claim holds for $m = 1$.

Assume then that $m \geq 2$. Denote $\epsilon' := \frac{\epsilon}{\sum_{l=0}^{m-2} 2^l (1-\kappa_p)^l} > 0$. Notice that ϵ' only depends on ϵ, m, p and n. Our strategy is to iterate Proposition 7.3.1 $m-1$ times with ϵ'. The first usage will be to $A_p(1, B^{2^m}, 1, g)$. What do we mean by iterating Proposition 7.3.1? Using Proposition 7.3.1 results in a product and one of the terms is an A_p-term. We apply Proposition 7.3.1 again to that A_p-term. We continue in this manner and get a chain of inequalities. Let $C = C_{\epsilon,\nu,m,E,p,n} > 0$ be the implicit constant of Proposition 7.3.1 when using ϵ'.

Claim: For all $k \in \mathbb{N}$ it holds that: if $k \leq m-1$, then Proposition 7.3.1 applied k times to $A_p(1, B^{2^m}, 1, g)$ with ϵ' results in

$$C^{\sum_{l=0}^{k-1} (1-\kappa_p)^l} \delta^{-\epsilon' \sum_{l=0}^{k-1} 2^l (l-\kappa_p)^l} A_p(2^k, B^{2^m}, 2^k, g)^{(1-\kappa_p)^k} \prod_{l=0}^{k-1} D_p(2^l, B^{2^m}, g)^{\kappa_p (1-\kappa_p)^l}. \tag{7.9}$$

We prove this by induction. Clearly (7.9) holds for $k = 0$.

Then assume that (7.9) holds for some k. If $k \geq m-1$, then there is nothing to prove. Assume then that $k \leq m-2$. The proposition applied k times results in (7.9) by our assumption. Then by applying Proposition (7.3.1) we get

$$\begin{aligned}
&A_p(2^k, B^{2^m}, 2^k, g)^{(1-\kappa_p)^k} \\
&\leq C^{(1-\kappa_p)^k} \delta^{-2^k \epsilon' (1-\kappa_p)^k} A_p(2^{k+1}, B^{2^m}, 2^{k+1}, g)^{(1-\kappa_p)(1-\kappa_p)^k} D_p(2^k, B^{2^m}, g)^{\kappa_p (1-\kappa_p)^k}
\end{aligned}$$

Plugging this estimate into (7.9) gives (7.9) for $k+1$.

Thus by induction the *Claim* holds. Especially it holds for $k = m-1$ and we get

$$\begin{aligned}
&A_p(1, B^{2^m}, 1, g) \\
&\lesssim_{\epsilon,\nu,m,E,p,n} \delta^{-\epsilon' \sum_{l=0}^{m-2} 2^l (1-\kappa_p)^l} A_p(2^{m-1}, B^{2^m}, 2^{m-1}, g)^{(1-\kappa_p)^{m-1}} \\
&\quad \cdot \prod_{l=0}^{m-2} D_p(2^l, B^{2^m}, g)^{\kappa_p (1-\kappa_p)^l} \\
&= \delta^{-\epsilon} A_p(2^{m-1}, B^{2^m}, 2^{m-1}, g)^{(1-\kappa_p)^{m-1}} \prod_{l=0}^{m-2} D_p(2^l, B^{2^m}, g)^{\kappa_p (1-\kappa_p)^l},
\end{aligned}$$

where the implicit constant is $C^{\sum_{l=0}^{m-2} (1-\kappa_p)^l}$, which only depends on ϵ, ν, m, E, p and n. $\square$

Chapter 8

Decoupling when $2 \le p \le \frac{2n}{n-1}$

We will use the results of Chapter 7 to dominate $\mathrm{Dec}_n(\delta, p, \nu, m, E)$ with a power of δ. Through the induction present in Theorem 6.2.1, we will then be able to dominate $\mathrm{Dec}_n(\delta, p, E)$ with a power of δ as well.

We continue to use the notations B^s, D_t, A_p and so on. These are familiar from Chapter 7.

8.1 A proof of $\mathrm{Dec}_n(\delta, p) \lesssim_{\epsilon,p,n} \delta^{-\epsilon}$ for $2 \le p \le \frac{2n}{n-1}$

We start this section with a lemma. Compare this with Definition 6.1.2.

Lemma 8.1.1. *Let $n \ge 2$, $2 \le p \le 20$ and $E \ge 100n$. Let $0 < \nu < 1$ and $m \in \mathbb{N}_+$. Let $Q_1, \dots, Q_n \subset [0,1]^{n-1}$ be ν-transverse cubes with equal side lengths $l(Q_i) = \mu$. Let $\delta \in 2^{-\mathbb{N}}$, $0 < \delta < 1$, be such that $\delta \le \mu$. Using the notation of Chapter 7 we have*

$$\Big[\sum_{\Delta \in \mathrm{Part}_{\mu^{-1}}(B^{2^m})} \Big(\prod_{i=1}^{n} \|E_{Q_i} g\|^p_{L^p(w_{\Delta,10E})} \Big)^{\frac{1}{n}} \Big]^{\frac{1}{p}}$$
$$\lesssim_{E,p,n} \delta^{-\frac{n-1}{2}} \delta^{-\frac{2^m n}{p}} A_p(1, B^{2^m}, 1, g)$$

for all cubes $B^{2^m} \subset \mathbb{R}^n$ and $g \in L^1([0,1]^{n-1})$.

Proof. We get by Cauchy-Schwarz and $\mu \le 1$ that

$$\begin{aligned}
&\Big[\delta^{2^m n} \sum_{\Delta \in \mathrm{Part}_{\mu^{-1}}(B^{2^m})} \Big(\prod_{i=1}^n \|E_{Q_i} g\|_{L^p(w_{\Delta,10E})}^p\Big)^{\frac{1}{n}}\Big]^{\frac{1}{p}} \\
&= \Big[\delta^{2^m n} \sum_{\Delta \in \mathrm{Part}_{\mu^{-1}}(B^{2^m})} \Big(\prod_{i=1}^n \Big\| \sum_{q_i \in \mathrm{Part}_\delta(Q_i)} E_{q_i} g\Big\|_{L^p(w_{\Delta,10E})}^p\Big)^{\frac{1}{n}}\Big]^{\frac{1}{p}} \\
&\le \Big[\delta^{2^m n} \sum_{\Delta \in \mathrm{Part}_{\mu^{-1}}(B^{2^m})} \Big(\prod_{i=1}^n \Big\| \sum_{q_i \in \mathrm{Part}_\delta(Q_i)} |E_{q_i} g|\Big\|_{L^p(w_{\Delta,10E})}^p\Big)^{\frac{1}{n}}\Big]^{\frac{1}{p}} \\
&\le \mu^{\frac{n-1}{2}} \delta^{-\frac{n-1}{2}} \Big[\delta^{2^m n} \sum_{\Delta \in \mathrm{Part}_{\mu^{-1}}(B^{2^m})} \Big(\prod_{i=1}^n \Big\| \Big(\sum_{q_i \in \mathrm{Part}_\delta(Q_i)} |E_{q_i} g|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(w_{\Delta,10E})}^p\Big)^{\frac{1}{n}}\Big]^{\frac{1}{p}} \\
&\le \delta^{-\frac{n-1}{2}} \Big[\delta^{2^m n} \sum_{\Delta \in \mathrm{Part}_{\mu^{-1}}(B^{2^m})} \Big(\prod_{i=1}^n \|h_i\|_{L^p(w_{\Delta,10E})}^p\Big)^{\frac{1}{n}}\Big]^{\frac{1}{p}},
\end{aligned}$$

where we denote

$$h_i := \Big(\sum_{q_i \in \mathrm{Part}_\delta(Q_i)} |E_{q_i} g|^2\Big)^{\frac{1}{2}}.$$

Then by n-Hölder and Lemma 4.1.5 we get

$$\begin{aligned}
&\sum_{\Delta \in \mathrm{Part}_{\mu^{-1}}(B^{2^m})} \Big(\prod_{i=1}^n \|h_i\|_{L^p(w_{\Delta,10E})}^p\Big)^{\frac{1}{n}} \\
&= \sum_{B^1 \in \mathrm{Part}_{\delta^{-1}}(B^{2^m})} \sum_{\Delta \in \mathrm{Part}_{\mu^{-1}}(B^1)} \Big(\prod_{i=1}^n \|h_i\|_{L^p(w_{\Delta,10E})}^p\Big)^{\frac{1}{n}} \\
&\le \sum_{B^1 \in \mathrm{Part}_{\delta^{-1}}(B^{2^m})} \prod_{i=1}^n \Big(\sum_{\Delta \in \mathrm{Part}_{\mu^{-1}}(B^1)} \|h_i\|_{L^p(w_{\Delta,10E})}^p\Big)^{\frac{1}{n}} \\
&= \sum_{B^1 \in \mathrm{Part}_{\delta^{-1}}(B^{2^m})} \prod_{i=1}^n \Big(\sum_{\Delta \in \mathrm{Part}_{\mu^{-1}}(B^1)} \int |h_i|^p w_{\Delta,10E}\Big)^{\frac{1}{n}} \\
&= \sum_{B^1 \in \mathrm{Part}_{\delta^{-1}}(B^{2^m})} \prod_{i=1}^n \Big(\int |h_i|^p \sum_{\Delta \in \mathrm{Part}_{\mu^{-1}}(B^1)} w_{\Delta,10E}\Big)^{\frac{1}{n}} \\
&\lesssim_{E,n} \sum_{B^1 \in \mathrm{Part}_{\delta^{-1}}(B^{2^m})} \prod_{i=1}^n \Big(\int |h_i|^p w_{B^1,10E}\Big)^{\frac{1}{n}} \\
&= \sum_{B^1 \in \mathrm{Part}_{\delta^{-1}}(B^{2^m})} \Big(\prod_{i=1}^n \|h_i\|_{L^p(w_{B^1,10E})}^p\Big)^{\frac{1}{n}}.
\end{aligned}$$

Combining these two inequalities we get

$$\Big[\delta^{2^m n} \sum_{\Delta \in \mathrm{Part}_{\mu^{-1}}(B^{2^m})} \Big(\prod_{i=1}^{n} \|E_{Q_i} g\|^p_{L^p(w_{\Delta,10E})}\Big)^{\frac{1}{n}}\Big]^{\frac{1}{p}}$$

$$\lesssim_{E,p,n} \delta^{-\frac{n-1}{2}} \Big[\delta^{2^m n} \sum_{B^1 \in \mathrm{Part}_{\delta^{-1}}(B^{2^m})} \Big(\prod_{i=1}^{n} \|h_i\|^p_{L^p(w_{B^1,10E})}\Big)^{\frac{1}{n}}\Big]^{\frac{1}{p}}.$$

Let $B^1 \in \mathrm{Part}_{\delta^{-1}}(B^{2^m})$ and $1 \leq i \leq n$. By Minkowski's inequality and the reverse Hölder inequality (Theorem 4.4.3) we have

$$\begin{aligned}
\|h_i\|^p_{L^p(w_{B^1,10E})} &= \Big\|\Big(\sum_{q_i \in \mathrm{Part}_\delta(Q_i)} |E_{q_i} g|^2\Big)^{\frac{1}{2}}\Big\|^p_{L^p(w_{B^1,10E})} \\
&= \Big\|\sum_{q_i \in \mathrm{Part}_\delta(Q_i)} |E_{q_i} g|^2\Big\|^{\frac{p}{2}}_{L^{\frac{p}{2}}(w_{B^1,10E})} \\
&\leq \Big(\sum_{q_i \in \mathrm{Part}_\delta(Q_i)} \||E_{q_i} g|^2\|_{L^{\frac{p}{2}}(w_{B^1,10E})}\Big)^{\frac{p}{2}} \\
&= \Big(\sum_{q_i \in \mathrm{Part}_\delta(Q_i)} \|E_{q_i} g\|^2_{L^p(w_{B^1,10E})}\Big)^{\frac{p}{2}} \\
&\lesssim_{E,p,n} \Big(\sum_{q_i \in \mathrm{Part}_\delta(Q_i)} |B^1|^{\frac{2}{p}} \|E_{q_i} g\|^2_{L^2_\#(w_{B^1,\frac{20E}{p}})}\Big)^{\frac{p}{2}} \\
&= \delta^{-n}\Big(\sum_{q_i \in \mathrm{Part}_\delta(Q_i)} \|E_{q_i} g\|^2_{L^2_\#(w_{B^1,\frac{20E}{p}})}\Big)^{\frac{p}{2}} \\
&\leq \delta^{-n}\Big(\sum_{q_i \in \mathrm{Part}_\delta(Q_i)} \|E_{q_i} g\|^2_{L^2_\#(w_{B^1,E})}\Big)^{\frac{p}{2}}. \qquad (p \leq 20)
\end{aligned}$$

In the application of the reverse Hölder inequality it was essential that $1 \leq 2 \leq p$, $10E > \frac{np}{2}$, $\delta^{-1} \geq 1$ and $l(q_i) = (l(B^1))^{-1}$.

All together, we have

$$
\begin{aligned}
&\Big[\delta^{2^m n} \sum_{\Delta \in \mathrm{Part}_{\mu^{-1}}(B^{2^m})} \Big(\prod_{i=1}^n \|E_{Q_i} g\|^p_{L^p(w_{\Delta,10E})}\Big)^{\frac{1}{n}}\Big]^{\frac{1}{p}} \\
&\lesssim_{E,p,n} \delta^{-\frac{n-1}{2}} \Big[\delta^{2^m n} \sum_{B^1 \in \mathrm{Part}_{\delta^{-1}}(B^{2^m})} \Big(\prod_{i=1}^n \delta^{-n} \Big(\sum_{q_i \in \mathrm{Part}_\delta(Q_i)} \|E_{q_i} g\|^2_{L^2_\#(w_{B^1,E})}\Big)^{\frac{p}{2}}\Big)^{\frac{1}{n}}\Big]^{\frac{1}{p}} \\
&= \delta^{-\frac{n-1}{2}} \Big[\delta^{2^m n} \delta^{-n} \sum_{B^1 \in \mathrm{Part}_{\delta^{-1}}(B^{2^m})} \Big(\prod_{i=1}^n \Big(\sum_{q_i \in \mathrm{Part}_\delta(Q_i)} \|E_{q_i} g\|^2_{L^2_\#(w_{B^1,E})}\Big)^{\frac{1}{2}}\Big)^{\frac{p}{n}}\Big]^{\frac{1}{p}} \\
&= \delta^{-\frac{n-1}{2}} \Big[\delta^{2^m n} \delta^{-n} \sum_{B^1 \in \mathrm{Part}_{\delta^{-1}}(B^{2^m})} D_2(1, B^1, g)^p\Big]^{\frac{1}{p}} \\
&= \delta^{-\frac{n-1}{2}} \Big[\frac{1}{|\mathcal{B}_1(B^{2^m})|} \sum_{B^1 \in \mathcal{B}_1(B^{2^m})} D_2(1, B^1, g)^p\Big]^{\frac{1}{p}} \\
&= \delta^{-\frac{n-1}{2}} A_p(1, B^{2^m}, 1, g).
\end{aligned}
$$

□

Lemma 8.1.2. *Let $n \geq 2$, $2 \leq p \leq \frac{2n}{n-1}$, $E \geq 100n$, $\delta \in 2^{-\mathbb{N}}$ such that $0 < \delta < 1$, $\epsilon > 0$, $0 < \nu < 1$ and $m \in \mathbb{N}_+$. Let $Q_1, \dots, Q_n \subset [0,1]^{n-1}$ be ν-transverse cubes with equal side lengths $l(Q_i) = \mu \geq \delta$. Then*

$$\delta^{-\frac{2^m n}{p}} A_p(1, B^{2^m}, 1, g) \lesssim_{\epsilon,\nu,m,E,p,n} \delta^{-\epsilon} \Big[\prod_{i=1}^n \sum_{q_i \in \mathrm{Part}_{\delta^{2^m-1}}(Q_i)} \|E_{q_i} g\|^2_{L^p(w_{B^{2^m},E})}\Big]^{\frac{1}{2n}}.$$

for all cubes $B^{2^m} \subset \mathbb{R}^n$ and $g \in L^1([0,1]^{n-1})$.

Proof. Now $\kappa_p = 0$. By Proposition 7.3.2, we can write

$$A_p(1, B^{2^m}, 1, g) \lesssim_{\epsilon,\nu,m,E,p,n} \delta^{-\epsilon} A_p(2^{m-1}, B^{2^m}, 2^{m-1}, g).$$

Furthermore by Corollary 7.2.4 we get

$$A_p(2^{m-1}, B^{2^m}, 2^{m-1}, g) \lesssim_{E,p,n} D_p(2^{m-1}, B^{2^m}, g).$$

Furthermore,

$$\delta^{-\frac{2^m n}{p}} D_p(2^{m-1}, B^{2^m}, g) = \Big[\prod_{i=1}^n \sum_{q_i \in \mathrm{Part}_{\delta^{2^m-1}}(Q_i)} \|E_{q_i} g\|^2_{L^p(w_{B^{2^m},E})}\Big]^{\frac{1}{2n}},$$

which follows easily by removing the normalization (#) from D_p. The claim follows after applying these estimates. □

Here is a reminder of what we are about to prove:

Definition 8.1.3. (**Decoupling constant**). Let $n \geq 2$, $p \geq 2$ and $\delta \in 4^{-\mathbb{N}}$. We define the decoupling constant $\mathrm{Dec}_n(\delta, p, 100n)$ to be the smallest non-negative real number such that

$$\|Eg\|_{L^p(w_{B,100n})} \leq \mathrm{Dec}_n(\delta, p, 100n)\Big(\sum_{Q \in \mathrm{Part}_{\delta^{\frac{1}{2}}}([0,1]^{n-1})} \|E_Q g\|^2_{L^p(w_{B,100n})}\Big)^{\frac{1}{2}}$$

holds for every cube $B \subset \mathbb{R}^n$ with side length δ^{-1} and every $g : [0,1] \to \mathbb{C}$.

Theorem 8.1.4. (Decoupling for $2 \leq p \leq \frac{2n}{n-1}$.) *Let us fix $n \in \mathbb{N}, n \geq 2$ and $2 \leq p \leq \frac{2n}{n-1}$. In addition, let us fix $\epsilon > 0$. Now there exists a constant $C > 0$ such that the following statement holds:*

$$\mathrm{Dec}_n(\delta, p, 100n) \leq C \cdot \delta^{-\epsilon}$$

for all $\delta \in 4^{-\mathbb{N}}$. That is,

$$\mathrm{Dec}_n(\delta, p, 100n) \lesssim_{\epsilon,p,n} \delta^{-\epsilon}$$

for all $\delta \in 4^{-\mathbb{N}}$.

We will prove this for all $E \geq 100n$, wherein the implicit constant will additionally depend on E. Of course, if $E = 100n$, then E is completely determined by n.

Proof. (*Proof of Theorem 8.1.4*)
Let $n \in \mathbb{N}$, $n \geq 2$. Assume $2 \leq p \leq \frac{2n}{n-1}$. Let $E \geq 100n$. Let $\epsilon > 0$.

Let $m \in \mathbb{N}_+$. Let $\mathbb{N}_m := \{2^{m-1}k \colon k \in \mathbb{N}\}$. Let $\delta \in 4^{-\mathbb{N}_m}$ be such that $0 < \delta < 1$. Let $0 < \nu < 1$. Let $Q_1, \dots, Q_n \subset [0,1]^{n-1}$ be ν-transverse cubes with equal side lengths μ satisfying $\mu \geq \delta^{2^{-m}}$. Let $B^1 \subset \mathbb{R}^n$ be a cube with side length δ^{-1} and let $g \in L^1([0,1]^{n-1})$.

Denote $\delta' := \delta^{2^{-m}}$. Our assumption about δ guarantees that $\delta' \in 2^{-\mathbb{N}}$ and $0 < \delta' < 1$. Also, $\delta' \leq \mu$ and $p \leq 20$. Denote $\epsilon' = 2^{m-2}\epsilon$. In what follows, the extra apostrophe (e.g. in A'_p) means that the quantity is relative to δ' instead of δ. Now Lemma 8.1.1 combined with Lemma 8.1.2 gives

$$\begin{aligned}
&\Big[\sum_{\Delta \in \mathrm{Part}_{\mu^{-1}}((B^{2^m})')} \Big(\prod_{i=1}^n \|E_{Q_i} g\|^p_{L^p(w_{\Delta,10E})}\Big)^{\frac{1}{n}}\Big]^{\frac{1}{p}} \\
&\lesssim_{E,p,n} (\delta')^{-\frac{n-1}{2}} (\delta')^{-\frac{2^m n}{p}} A'_p(1, (B^{2^m})', 1, g) \\
&\lesssim_{\epsilon,\nu,m,E,p,n} (\delta')^{-\frac{n-1}{2}} (\delta')^{-\epsilon'} \Big[\prod_{i=1}^n \sum_{q_i \in \mathrm{Part}_{(\delta')^{2^{m-1}}}(Q_i)} \|E_{q_i} g\|^2_{L^p(w_{(B^{2^m})',E})}\Big]^{\frac{1}{2n}}.
\end{aligned}$$

Substituting $\delta' = \delta^{2^{-m}}$ gives

$$\Big[\sum_{\Delta \in \mathrm{Part}_{\mu^{-1}}(B^1)} \Big(\prod_{i=1}^{n} \|E_{Q_i} g\|^p_{L^p(w_{\Delta,10E})}\Big)^{\frac{1}{n}}\Big]^{\frac{1}{p}}$$
$$\lesssim_{\epsilon,\nu,m,E,p,n} \delta^{-(n-1)2^{-m-1}} \delta^{-\frac{\epsilon}{4}} \Big[\prod_{i=1}^{n} \sum_{q_i \in \mathrm{Part}_{\delta^{\frac{1}{2}}}(Q_i)} \|E_{q_i} g\|^2_{L^p(w_{B^1,E})}\Big]^{\frac{1}{2n}}.$$

Thus by Definition 6.1.2

$$\mathrm{Dec}_n(\delta, p, \nu, m, E) \lesssim_{\epsilon,\nu,m,E,p,n} \delta^{-(n-1)2^{-m-1}} \delta^{-\frac{\epsilon}{4}} \tag{8.1}$$

for all $\delta \in 4^{-\mathbb{N}_m}$. Note that $1 \in 4^{-\mathbb{N}_m}$. Above we particularly assumed that $\delta < 1$. However, the inequality (8.1) in the case $\delta = 1$ follows from the estimate $\mathrm{Dec}_n(\delta, p, \nu, m, E) \le \delta^{-(n-1)/4}$ (see (6.1)).

From this point onward, we fix $m = m(\epsilon, n) \in \mathbb{N}_+$ large enough so that $(n-1)2^{-m} \le \epsilon/4$ and $m \ge 2$. Let $\delta \in 4^{-\mathbb{N}}$ be such that $\delta \le 4^{-2^m}$. By Proposition 6.1.4 there is $\delta_1 \in 4^{-\mathbb{N}_m}$ such that $\delta \le \delta_1$ and

$$\mathrm{Dec}_n(\delta, p, \nu, m, E) \lesssim_{\epsilon,E,p,n} \mathrm{Dec}_n(\delta_1, p, \nu, m-1, E).$$

Observe that $\delta_1 \in 4^{-\mathbb{N}_{m-1}}$. Thus by (8.1) we get

$$\mathrm{Dec}_n(\delta_1, p, \nu, m-1, E) \lesssim_{\epsilon,\nu,E,p,n} \delta_1^{-(n-1)2^{-m}} \delta_1^{-\frac{\epsilon}{4}} \le \delta_1^{-\frac{\epsilon}{4}} \delta_1^{-\frac{\epsilon}{4}} = \delta_1^{-\frac{\epsilon}{2}} \le \delta^{-\frac{\epsilon}{2}}.$$

Combining these two inequalities we have

$$\mathrm{Dec}_n(\delta, p, \nu, m, E) \lesssim_{\epsilon,\nu,E,p,n} \delta^{-\frac{\epsilon}{2}} \tag{8.2}$$

for all $\delta \in 4^{-\mathbb{N}}$ such that $\delta \le 4^{-2^m}$. Then assume that $\delta \in 4^{-\mathbb{N}}$ and $\delta > 4^{-2^m}$. Then by (6.1)

$$\mathrm{Dec}_n(\delta, p, \nu, m, E) \le \delta^{-\frac{n-1}{4}} \le 4^{\frac{2^m(n-1)}{4}} \lesssim_{\epsilon,n} 1 \le \delta^{-\frac{\epsilon}{2}}.$$

We just proved that for *all* $\delta \in 4^{-\mathbb{N}}$

$$\mathrm{Dec}_n(\delta, p, \nu, m, E) \lesssim_{\epsilon,\nu,E,p,n} \delta^{-\frac{\epsilon}{2}}, \tag{8.3}$$

where $m = m(\epsilon, n)$.

We proceed to use induction on n and Theorem 6.2.1.

First, let $n = 2$. Recall that we use $m = m(\epsilon, n)$. Let $\theta = \theta_{E,p,n}$ be the function described in Theorem 6.2.1. Let $\nu = \nu(\epsilon, E, p, n)$ be such that $0 < \theta(\nu) <$

$\min\{\epsilon/2, 1\}$. Now there exist constants $C = C_{\nu,m} > 0$ and $D = D_{\nu,m} > 0$ such that when $\delta \in 4^{-\mathbb{N}}$ and $\delta \le D^{-1}$, we have

$$\begin{aligned}
\mathrm{Dec}_n(\delta, p, E) &\le C\delta^{-\theta(\nu)}\Big(1 + \sup_{\delta \le \delta' \le 1} \mathrm{Dec}_n(\delta', p, \nu(\epsilon, E, p, n), m(\epsilon, n), E)\Big) \\
&\lesssim_{\epsilon,E,p,n} \delta^{-\theta(\nu)}\Big(1 + \sup_{\delta \le \delta' \le 1} \mathrm{Dec}_n(\delta', p, \nu(\epsilon, E, p, n), m(\epsilon, n), E)\Big) \\
&\overset{(8.3)}{\lesssim_{\epsilon,E,p,n}} \delta^{-\theta(\nu)}\Big(1 + \sup_{\delta \le \delta' \le 1} (\delta')^{-\frac{\epsilon}{2}}\Big) \\
&\le \delta^{-\theta(\nu)}\Big(1 + \delta^{-\frac{\epsilon}{2}}\Big) \\
&\lesssim \delta^{-\theta(\nu)}\delta^{-\frac{\epsilon}{2}} \\
&\le \delta^{-\frac{\epsilon}{2}}\delta^{-\frac{\epsilon}{2}} \\
&= \delta^{-\epsilon}.
\end{aligned}$$

If $\delta \in 4^{-\mathbb{N}}$ and $\delta > D^{-1}$, then

$$\mathrm{Dec}_n(\delta, p, E) \le \delta^{-\frac{n-1}{4}} \le D^{\frac{n-1}{4}} \lesssim_{\epsilon,E,p,n} 1 \le \delta^{-\epsilon}.$$

Hence, when $n = 2$,

$$\mathrm{Dec}_n(\delta, p, E) \lesssim_{\epsilon,E,p,n} \delta^{-\epsilon}$$

for all $\delta \in 4^{-\mathbb{N}}$, $E \ge 100n$, $2 \le p \le \frac{2n}{n-1}$ and $\epsilon > 0$.

Then we do the induction step. Assume that $n \ge 3$ and

$$\mathrm{Dec}_{n-1}(\delta, p, E) \lesssim_{\epsilon,E,p,n} \delta^{-\epsilon} \tag{8.4}$$

for all $\delta \in 4^{-\mathbb{N}}$, $E \ge 100(n-1)$, $2 \le p \le \frac{2(n-1)}{n-2}$ and $\epsilon > 0$.

Fix $2 \le p \le \frac{2n}{n-1}$ and $E \ge 100n$. Then $2 \le p \le \frac{2(n-1)}{n-2}$. Also, $10E \ge 100(n-1)$ and thus $\Gamma_{n-1}(10E) \ge 100(n-1)$. Then (8.4) particularly gives

$$\mathrm{Dec}_{n-1}(\delta, p, \Gamma_{n-1}(10E)) \lesssim_{\epsilon,E,p,n} \delta^{-\epsilon}$$

for all $\delta \in 4^{-\mathbb{N}}$ and $\epsilon > 0$.

Thus the condition (ii) of Theorem 6.2.1 holds. Then we fix $\epsilon > 0$ as well and continue the proof as we did in the case $n = 2$. Ultimately we get

$$\mathrm{Dec}_n(\delta, p, E) \lesssim_{\epsilon,E,p,n} \delta^{-\epsilon}$$

for all $\delta \in 4^{-\mathbb{N}}$. □

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